Solving Ramanujan's Puzzling Problem

Evaluating an Infinite Sequence of Nested Radicals

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"An infinitely nested structure with ever-increasing coefficients collapses to the simple integer 2."

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1 Introduction

Here's a problem that looks impossible at first glance. You've got infinitely many nested square roots, each level multiplied by an increasing integer. How do you even begin to evaluate something like that?

Turns out, Ramanujan already solved this over a century ago—and the answer is surprisingly clean.

2 The Problem: An Infinite Sequence of Nested Radicals

Consider the following sequence of functions:

$$f_1(x) = \sqrt{1+\sqrt{x}} \tag{1}$$

$$f_2(x) = \sqrt{1+\sqrt{1+2\sqrt{x}}} \tag{2}$$

$$f_3(x) = \sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{x}}}} \tag{3}$$

$$f_4(x) = \sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{x}}}}} \tag{4}$$

See the pattern? Each function adds another layer of nesting, and the coefficient in front of the innermost radical increases: $1, 2, 3, 4, \ldots$

The general form is:

$$f_n(x) = \sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{\cdots\sqrt{1+n\sqrt{x}}}}}} \tag{5}$$

Problem 1. What happens as $n \to \infty$? That is, evaluate:

$$\lim_{n\to\infty} f_n(x) = \ ? \tag{6}$$

At first, this seems hopeless. You can't "start from the inside" because there's always another layer. And unlike simple recursive sequences, each level has a different coefficient.

But Ramanujan cracked this over a century ago.

3 The Key Insight: Ramanujan's Nested Radical Formula

The solution relies on Ramanujan's general formula for nested radicals.

Theorem 3.1 (Ramanujan's Nested Radical Identity)**.** For real numbers x, n, a:

$$x+n+a = \sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\cdots}}} \tag{7}$$

This formula tells us that certain infinite nested radicals have clean, closed-form values. The trick is matching our problem to this template.

4 Derivation of the Solution

4.1 Step 1: Derive a Useful Special Case

Let's set $n = 1$ and $a = 0$ in Ramanujan's formula:

$$x + 1 = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + (x+3)\sqrt{\cdots}}}}} \tag{8}$$

Now here's the key move. Set $x = 2$:

$$2 + 1 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\cdots}}}}} \tag{9}$$

Which simplifies to:

Ramanujan's Famous Result (1911)

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\cdots}}}}} \tag{10}$$

This is one of Ramanujan's most celebrated identities. He posed it as a challenge in the *Journal of the Indian Mathematical Society* in 1911, and it took months before anyone could prove it.

4.2 Step 2: Connect to Our Problem

Now look at our original sequence again. As $n \to \infty$, what does $f_n(x)$ approach?

$$\lim_{n\to\infty} f_n(x) = \sqrt{1 + \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\cdots}}}}} \tag{11}$$

Notice anything? **The inner part is exactly Ramanujan's result!**
Let us define:

$$X = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\cdots}}}}} = 3 \tag{12}$$

Then our limit becomes:

$$\lim_{n\to\infty} f_n(x) = \sqrt{1 + X} = \sqrt{1 + 3} = \sqrt{4} = 2 \tag{13}$$

5 The Answer

Final Result

$$\lim_{n\to\infty} f_n(x) = 2 \tag{14}$$

Equivalently:

$$\sqrt{1 + \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\cdots}}}}} = 2 \tag{15}$$

That's it. An infinitely nested structure with ever-increasing coefficients collapses to the simple integer 2.

6 Numerical Verification

Let's verify this result by computing the first several values of $f_n(x)$ and checking convergence toward 2.

For simplicity, we evaluate with $x = 1$ (the limit doesn't actually depend on x, as the innermost term becomes negligible):

n	Expression	Approximate Value
1	$\sqrt{1+\sqrt{1}}$	1.4142
2	$\sqrt{1+\sqrt{1+2\sqrt{1}}}$	1.6529
3	$\sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1}}}}$	1.8174
4	$\sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1}}}}}$	1.9035
5	(5 levels deep)	1.9486
10	(10 levels deep)	1.9937
20	(20 levels deep)	1.9997
∞	Limit	**2.0000**

Table 1: Convergence of $f_n(1)$ to the limit 2

The convergence is clear. Each additional layer brings us closer to 2.

Remark 6.1. You can verify this yourself with a simple recursive program. Start with some value, apply $\sqrt{1 + n \cdot \text{(previous)}}$ working outward, and watch it converge.

7 Why Does x Disappear?

You might wonder: the original function $f_n(x)$ depends on x, but our answer is just 2. What happened?

Here's the intuition: as $n \to \infty$, the innermost term $n\sqrt{x}$ gets wrapped in so many layers of square roots that its influence becomes negligible.

Think about it this way:

- The innermost term is $n\sqrt{x}$
- After one square root: $\sqrt{1 + n\sqrt{x}} \approx \sqrt{n\sqrt{x}}$ for large n
- Each subsequent layer dampens the effect further
- By the time you reach the outermost layer, the initial x has been "washed out"

Mathematically, Ramanujan's formula shows that the infinite structure has a definite value regardless of how it terminates—the pattern of coefficients $(1, 2, 3, 4, \ldots)$ determines the limit completely.

8 Understanding the Nested Structure

Let me break down exactly how our problem maps to Ramanujan's formula:

Our Problem	**Ramanujan's Formula** ($x = 2, n = 1, a = 0$)
Outermost: $\sqrt{1 + \cdots}$	One extra layer wrapping $x + 1 = 3$
Coefficient 2	Corresponds to $x = 2$
Coefficient 3	Corresponds to $x + 1 = 3$
Coefficient 4	Corresponds to $x + 2 = 4$
Coefficient k	Corresponds to $x + (k - 2) = k$

Table 2: Mapping between our problem and Ramanujan's formula

The coefficients $2, 3, 4, 5, \ldots$ in our nested radical match the sequence $x, x + 1, x + 2, x + 3, \ldots$ in Ramanujan's formula when $x = 2$.

9 Related Results and Generalizations

Once you understand this technique, you can evaluate an entire family of nested radicals.

9.1 Variation 1: Different Starting Coefficient

What if the coefficients started at 3 instead of 2?

$$\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\cdots}}}} = ? \tag{16}$$

Using Ramanujan's formula with $x = 3$:

$$x + 1 = 3 + 1 = 4 \tag{17}$$

So this nested radical equals **4**.

9.2 Variation 2: The Classic Ramanujan Problem

Ramanujan's original 1911 challenge:

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\cdots}}}} = 3 \tag{18}$$

This is the inner part of our problem—we just added one more layer on top.

9.3 General Formula

Theorem 9.1 (General Pattern)**.** For coefficients starting at k:

$$\sqrt{1 + k\sqrt{1 + (k + 1)\sqrt{1 + (k + 2)\sqrt{\cdots}}}} = k + 1 \tag{19}$$

So the pattern is beautifully simple: **starting coefficient plus one**.

Proof. Set $x = k$ and $n = 1$, $a = 0$ in Ramanujan's formula:

$$x + 1 = \sqrt{1 + x\sqrt{1 + (x + 1)\sqrt{1 + (x + 2)\sqrt{\cdots}}}} \tag{20}$$

With $x = k$, the left side is $k + 1$, and the right side has coefficients $k, k + 1, k + 2, \ldots$ □

9.4 Adding Extra Layers

Our main result can be generalized:

Corollary 9.2. Adding m extra layers of $\sqrt{1+\cdots}$ on top of Ramanujan's result:

$$\underbrace{\sqrt{1+\sqrt{1+\cdots\sqrt{1+}}}}_{m \text{ extra layers}}\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}} \tag{21}$$

converges to the value obtained by starting with 3 and iterating $x \mapsto \sqrt{1+x}$ exactly m times.

For $m = 1$: $\sqrt{1+3} = 2$
For $m = 2$: $\sqrt{1+2} = \sqrt{3} \approx 1.732$
For $m = 3$: $\sqrt{1+\sqrt{3}} \approx 1.653$

10 Rigorous Convergence Analysis

Theorem 10.1 (Convergence)**.** For any $x > 0$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges, and:

$$\lim_{n\to\infty} f_n(x) = 2 \tag{22}$$

Proof Sketch. The rigorous proof involves three steps:

Step 1: Monotonicity. Show that for each fixed $x > 0$, the sequence $f_n(x)$ is monotonically increasing. This follows from the fact that adding deeper nesting with positive coefficients increases the value.

Step 2: Boundedness. Show that $f_n(x) < 2$ for all finite n. This can be established by induction, using the fact that if $f_{n-1}(x) < 3$, then $f_n(x) < \sqrt{1+3} = 2$.

Step 3: Identification of Limit. By the Monotone Convergence Theorem, since $\{f_n(x)\}$ is monotonically increasing and bounded above, the limit exists. Ramanujan's identity then establishes that this limit equals 2. □

11 Historical Context

Ramanujan posed the identity

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}} \tag{23}$$

as Question 289 in the *Journal of the Indian Mathematical Society* in 1911. He was just 23 years old.

The problem remained unsolved for several months until Ramanujan himself provided the solution. His approach used the binomial expansion and clever algebraic manipulation—the same techniques that led to his general nested radical formula.

What makes this result remarkable is not just the answer, but the fact that Ramanujan could "see" that such infinite structures had clean closed forms. His intuition for patterns in seemingly chaotic mathematical expressions was extraordinary.

G.H. Hardy later wrote that Ramanujan's formulas "defeated me completely; I had never seen anything in the least like them before."

12 Summary of Key Results

Main Results:

1. $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}} = 3$ (Ramanujan, 1911)
2. $\sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}}} = 2$ (Our problem)
3. General pattern: Starting coefficient k gives result $k+1$
4. Convergence is fast: 10 levels gets within 0.01 of the limit
5. The limit is independent of the starting value x

13 Further Reading

For the complete derivation of Ramanujan's general nested radical formula from the binomial theorem, see my companion monograph *Elementary Analysis on Ramanujan's Nested Radicals*, which covers:

- The full derivation starting from $(a+b)^2 = a^2 + 2ab + b^2$
- Convergence theory using Herschfeld's theorem
- The connection between nested radicals and continued fractions
- The golden ratio as the simplest nested radical
- Calculus of functions defined by nested radicals

These problems showcase what made Ramanujan special: the ability to see elegant patterns in seemingly chaotic mathematical structures—and to prove that infinity, properly tamed, gives finite answers.

References

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