Elementary Analysis on Ramanujan's Nested Radicals

A Mathematical Exploration

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"An equation for me has no meaning unless it expresses a thought of God."
— Srinivasa Ramanujan

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Abstract

Srinivasa Ramanujan's contributions to mathematics continue to inspire mathematicians worldwide. Among his earliest and most elegant discoveries are his formulas involving nested radicals—expressions where square roots are embedded within square roots, potentially infinitely deep. This monograph presents a comprehensive elementary analysis of Ramanujan's nested radicals, beginning with fundamental definitions and progressing through convergence proofs, connections to continued fractions, and applications in calculus.

We derive the general formula for Ramanujan's nested radicals from first principles using the binomial theorem, establish convergence criteria using Herschfeld's theorem, and explore the beautiful relationship between nested radicals and continued fractions. The document concludes with an examination of the differentiability and integrability of functions defined by nested radicals.

Keywords: Nested radicals, Ramanujan, continued fractions, golden ratio, convergence, calculus of radicals

1 Introduction and Historical Context

1.1 Srinivasa Ramanujan: A Brief Biography

Full Name	Srinivasa Iyengar Ramanujan
Born	December 22, 1887; Erode, Madras Presidency
Died	April 26, 1920 (aged 32); Kumbakonam, Madras Presidency
Education	Government Arts College; Pachaiyappa's College
Known for	Ramanujan prime, mock theta functions, Rogers–Ramanujan identities, Ramanujan's sum

Ramanujan was a self-taught mathematical genius who, despite having almost no formal training in pure mathematics, made substantial contributions to mathematical analysis, number theory, infinite series, and continued fractions. His notebooks, containing nearly 3,900 results, continue to be a source of research problems for mathematicians today.

1.2 Nested Radicals in Mathematical History

Nested radicals appear throughout the history of mathematics. The ancient Babylonians used iterative methods equivalent to nested radicals for computing square roots. Viète (1593) discovered the remarkable identity:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots \tag{1}$$

which expresses π as an infinite product of nested radicals.

Ramanujan's work elevated nested radicals from curiosities to a systematic theory, revealing deep connections to continued fractions, special functions, and algebraic number theory.

2 Fundamentals of Nested Radicals

2.1 Definitions and Notation

Definition 2.1 (Nested Radical). A **nested radical** is an expression of the form

$$\sqrt{x_1 + x_2\sqrt{x_3 + x_4\sqrt{x_5 + x_6\sqrt{x_7 + x_8\sqrt{\cdots}}}}} \tag{2}$$

where $x_i \in \mathbb{R}$ for all $i \in \mathbb{N}$.

The nesting can be finite or infinite:

- A **finite nested radical** has a definite number of radical signs, such as $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4}}}}$.
- An **infinite nested radical** extends indefinitely, such as $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots}}}}$.

Definition 2.2 (Depth of Nesting). The **depth** of a finite nested radical is the number of radical signs it contains. The depth of an infinite nested radical is ∞.

Remark 2.1. While we focus primarily on square roots, nested radicals can involve roots of any degree. For example,

$$\sqrt[4]{5+\sqrt[3]{11+\sqrt{23}}} \tag{3}$$

is a nested radical with mixed degrees. Such expressions arise naturally in the theory of solvability of polynomial equations by radicals.

2.2 Denesting: Simplifying Nested Radicals

Definition 2.3 (Denesting)**.** The process of reducing the depth of a nested radical is called **denesting**. A nested radical is **denestable** if it can be expressed using radicals of smaller depth.

Example 2.1. The expression $\sqrt{3+2\sqrt{2}}$ can be denested:

$$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2} \tag{4}$$

This can be verified by squaring: $(1+\sqrt{2})^2 = 1+2\sqrt{2}+2 = 3+2\sqrt{2}$.

2.2.1 Denesting Two-Level Radicals

Consider a nested radical of the form $\sqrt{a+\sqrt{b}}$ where $a, b \in \mathbb{Q}$. We seek to express it as $\sqrt{d}+\sqrt{e}$ where $d, e \in \mathbb{Q}$.

Theorem 2.1 (Two-Level Denesting)**.** Let $a, b \in \mathbb{Q}$ with $b > 0$. If there exist $d, e \in \mathbb{Q}$ such that

$$\sqrt{a+\sqrt{b}} = \sqrt{d}+\sqrt{e} \tag{5}$$

then d and e satisfy:

$$d+e = a \tag{6}$$

$$4de = b \tag{7}$$

Proof. Squaring both sides of the assumed equality:

$$a+\sqrt{b} = d+e+2\sqrt{de} \tag{8}$$

Equating rational and irrational parts yields equations (6) and (7). □

From these equations, we obtain the quadratic:

$$4e^2 - 4ae + b = 0 \tag{9}$$

with solutions:

$$e = \frac{a \pm \sqrt{a^2-b}}{2} \tag{10}$$

Denesting Criterion: The radical $\sqrt{a+\sqrt{b}}$ is denestable over $\mathbb{Q}$ if and only if a^2-b is a perfect square in $\mathbb{Q}$.

Example 2.2. To denest $\sqrt{5+2\sqrt{6}}$:

- Here $a = 5$ and $b = 24$ (since $2\sqrt{6} = \sqrt{24}$).
- Check: $a^2 - b = 25 - 24 = 1$, which is a perfect square.
- Solving: $e = \frac{5\pm1}{2}$, giving $e = 3$ or $e = 2$.
- Therefore: $\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$.

2.2.2 Denesting Three-Level Radicals

For $\sqrt{a + \sqrt{b + \sqrt{c}}}$, we seek $\sqrt{f} + \sqrt{g + \sqrt{h}}$. This leads to a more complex system:

$$a = f + g \tag{11}$$
$$b = h + 4fg \tag{12}$$
$$c = 16f^2gh + 16fg^2h + 16fgh^2 + 32f^2g\sqrt{h} + h^2\sqrt{h} \tag{13}$$

The algebraic complexity grows rapidly with depth. Susan Landau developed algorithms for denesting such expressions, which fall under the domain of computational algebra.

3 Ramanujan's General Formula for Nested Radicals

3.1 Derivation from the Binomial Theorem

We now derive Ramanujan's fundamental formula for nested radicals, starting from the elementary binomial expansion.

Theorem 3.1 (Ramanujan's Nested Radical Identity). For all $x, n, a \in \mathbb{R}$ with appropriate sign conditions:

$$x + n + a = \sqrt{ax + (n + a)^2 + x\sqrt{a(x + n) + (n + a)^2 + (x + n)\sqrt{a(x + 2n) + (n + a)^2 + \cdots}}} \tag{14}$$

Proof. We proceed by constructing a telescoping pattern.

Step 1: Begin with the algebraic identity

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{15}$$

Substituting $b = n + a$:

$$(x + n + a)^2 = x^2 + 2x(n + a) + (n + a)^2 \tag{16}$$

Rearranging:

$$(x + n + a)^2 = x^2 + (n + a)^2 + x(2n + 2a) \tag{17}$$

Factor the last term:

$$(x + n + a)^2 = x^2 + (n + a)^2 + x \cdot 2(n + a) \tag{18}$$

Rewrite using $ax + x \cdot 2n + xa = x(a + 2n + a) = x(2n + 2a)$:

$$(x + n + a)^2 = ax + (n + a)^2 + x(x + 2n + a) \tag{19}$$

Step 2: Taking the positive square root of (19):

$$x + n + a = \sqrt{ax + (n+a)^2 + x(x + 2n + a)} \tag{20}$$

Step 3: Replace x with $x + n$ in (20):

$$x + 2n + a = \sqrt{a(x+n) + (n+a)^2 + (x+n)(x + 3n + a)} \tag{21}$$

Step 4: Substitute (21) into (20):

$$x + n + a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)(x + 3n + a)}} \tag{22}$$

Continuing this process inductively:

$$x + kn + a = \sqrt{a(x + (k-1)n) + (n+a)^2 + (x + (k-1)n)(x + (k+1)n + a)} \tag{23}$$

This generates the infinite nested radical in (14). □

3.2 The General k-Term Formula

For finite k, the nested radical takes the explicit form:

$$x + n + a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\cdots + (x + (k-2)n)\sqrt{R_k}}}} \tag{24}$$

where the innermost term is:

$$R_k = a(x + (k-1)n) + (n+a)^2 + (x + (k-1)n)(x + (k+1)n + a) \tag{25}$$

4 Special Cases and Notable Identities

4.1 The Identity for 3

Setting $x = n = a = 1$ in Ramanujan's formula:

Theorem 4.1.

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\cdots}}}}} \tag{26}$$

Proof. With $x = n = a = 1$:

- Left side: $x + n + a = 1 + 1 + 1 = 3$
- First term: $ax + (n+a)^2 = 1 \cdot 1 + (1+1)^2 = 1 + 4 = 5$
- Second term coefficient: $x = 1$
- Under second radical: $a(x+n) + (n+a)^2 = 1 \cdot 2 + 4 = 6$, with coefficient $x + n = 2$
- Pattern continues: coefficients are $1, 2, 3, 4, \ldots$ and inner terms are $5, 6, 7, 8, \ldots$

□

An equivalent form, derived by adjusting parameters:

$$x + 1 = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + (x+3)\sqrt{\cdots}}}}} \tag{27}$$

Setting $x = 2$ recovers the formula for 3.

4.2 The Trivial Case: $n = a = 0$

When $n = 0 = a$ with $x \neq 0$:

$$x = \sqrt{x\sqrt{x\sqrt{x\sqrt{\cdots}}}} \tag{28}$$

This infinite nested radical equals x itself. To see why:

$$\sqrt{x \cdot \sqrt{x \cdot \sqrt{x \cdots}}} = x^{1/2} \cdot x^{1/4} \cdot x^{1/8} \cdots = x^{1/2+1/4+1/8+\cdots} = x^1 = x \tag{29}$$

More generally:

$$\underbrace{\sqrt{x\sqrt{x\sqrt{x\cdots\sqrt{x}}}}}_{k \text{ radicals}} = x^{1-2^{-k}} \tag{30}$$

As $k \to \infty$, this approaches x.

4.3 Nested Radicals for Cube of Integers

Setting $x = a = n$ yields:

$$3n = \sqrt{n^2 + 4n^2 + n\sqrt{4n^2 + 4n^2 + 2n\sqrt{9n^2 + 4n^2 + 3n\sqrt{\cdots}}}} \tag{31}$$

Simplifying:

$$3n = \sqrt{5n^2 + n\sqrt{8n^2 + 2n\sqrt{13n^2 + 3n\sqrt{\cdots}}}} \tag{32}$$

5 The Constant-Coefficient Case: Golden Ratio and Beyond

5.1 Nested Radicals of Type $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$

Consider the simplest infinite nested radical where all coefficients equal a:

$$L = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{\cdots}}}} \tag{33}$$

Theorem 5.1. For $a \geq 0$, the infinite nested radical $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ converges to:

$$L = \frac{1 + \sqrt{1 + 4a}}{2} \tag{34}$$

Proof. Assuming convergence to L, we have:

$$L = \sqrt{a + L} \tag{35}$$

Squaring: $L^2 = a + L$, which gives $L^2 - L - a = 0$.

By the quadratic formula:

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2} \tag{36}$$

Since L must be positive (as a limit of positive terms), we take the positive root. □

5.2 The Golden Ratio

The most celebrated special case occurs when $a = 1$:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots \tag{37}$$

where φ is the **golden ratio**.

The golden ratio satisfies $\varphi^2 = \varphi + 1$, which is precisely the equation we derived.

5.3 Connection to the Plastic Constant

For cubic nested radicals of the form:

$$P = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}} \tag{38}$$

The limit satisfies $P^3 = 1 + P$, giving:

$$P = \frac{1}{3}\left(1 + \sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}}\right) \approx 1.3247179572\ldots \tag{39}$$

This is the **plastic constant**, which appears in architecture and number theory.

6 Convergence Theory

6.1 Herschfeld's Theorem

The convergence of infinite nested radicals is governed by a beautiful theorem due to Aaron Herschfeld (1935).

Theorem 6.1 (Herschfeld's Theorem)**.** The infinite nested radical

$$\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\sqrt{a_4+\cdots}}}} \tag{40}$$

with $a_n \geq 0$ for all n converges if and only if

$$\limsup_{n\to\infty} a_n^{2^{-n}} < \infty \tag{41}$$

Corollary 6.2. If $a_n = O(n^k)$ for some constant k, then the nested radical converges.

Proof. For polynomial growth, $a_n \leq Cn^k$ for some $C > 0$. Then:

$$a_n^{2^{-n}} \leq (Cn^k)^{2^{-n}} = C^{2^{-n}} \cdot n^{k\cdot 2^{-n}} \tag{42}$$

As $n \to \infty$: $C^{2^{-n}} \to 1$ and $n^{k\cdot 2^{-n}} \to 1$. Thus $\limsup a_n^{2^{-n}} \leq 1 < \infty$. □

Example 6.1. The nested radical $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots}}}}$ converges because $a_n = n$ satisfies Herschfeld's condition.

6.2 Convergence of n-th Degree Nested Radicals

For nested radicals involving n-th roots:

$$L = \sqrt[n]{a + \sqrt[n]{a + \sqrt[n]{a + \cdots}}} \tag{43}$$

Theorem 6.3. The n-th degree constant nested radical converges to the unique positive root of:

$$L^n - L - a = 0 \tag{44}$$

When $n = 2$, this reduces to $L^2 - L - a = 0$, confirming our earlier result.

7 Nested Radicals and Continued Fractions

7.1 A Remarkable Equivalence

There exists a profound connection between nested radicals and continued fractions.

Theorem 7.1 (Nested Radical–Continued Fraction Duality). For $a, b \geq 0$:

$$\sqrt{a + b\sqrt{a + b\sqrt{a + b\sqrt{\cdots}}}} = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cdots}}} \tag{45}$$

Both expressions equal:

$$\frac{a + \sqrt{a^2 + 4b}}{2} \tag{46}$$

Proof. Let L denote the nested radical. Then $L = \sqrt{a + bL}$, giving $L^2 = a + bL$, hence:

$$L = \frac{b + \sqrt{b^2 + 4a}}{2} \tag{47}$$

Let C denote the continued fraction. Then $C = a + \frac{b}{C}$, giving $C^2 = aC + b$, hence:

$$C = \frac{a + \sqrt{a^2 + 4b}}{2} \tag{48}$$

Interchanging a and b in the nested radical formula yields the continued fraction formula. Both equal $\frac{a+\sqrt{a^2+4b}}{2}$ when the parameters are appropriately matched. □

7.2 The Golden Ratio Connection

Setting $a = b = 1$:

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} = \frac{1 + \sqrt{5}}{2} = \varphi \tag{49}$$

This demonstrates that the golden ratio can be represented both as an infinite nested radical and as the simplest infinite continued fraction $[1; 1, 1, 1, \ldots]$.

8 Ramanujan's Ultimate Nested Radical Formula

Ramanujan discovered another elegant formula expressing sums as nested radicals.

Theorem 8.1 (Ramanujan's Sum Formula). For any $x, a \in \mathbb{R}$:

$$x + a = \sqrt{a^2 + x\sqrt{a^2 + (x+a)\sqrt{a^2 + (x+2a)\sqrt{a^2 + (x+3a)\sqrt{\cdots}}}}} \tag{50}$$

Proof. Starting from:

$$(x+a)^2 = x^2 + a^2 + 2ax \tag{51}$$

Rearranging:

$$(x+a)^2 = a^2 + x(x+2a) \tag{52}$$

Taking square roots:

$$x + a = \sqrt{a^2 + x(x+2a)} \tag{53}$$

Replacing x by $x + a$ in (53):

$$x + 2a = \sqrt{a^2 + (x+a)(x+3a)} \tag{54}$$

Substituting back and continuing inductively yields the infinite nested radical. □

8.1 Generalizing to n Terms

$$x + a = \sqrt{a^2 + x\sqrt{a^2 + (x+a)\sqrt{a^2 + (x+2a)\sqrt{\cdots\sqrt{a^2 + (x+na)(x+(n+1)a)}}}}} \tag{55}$$

9 Ramanujan's Wild Theorem

In his famous letter to G.H. Hardy on January 16, 1913, Ramanujan included this remarkable identity:

Ramanujan's Wild Theorem:

$$\cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}} \tag{56}$$

This identity connects an infinite continued fraction with exponential terms to a finite expression involving the golden ratio and fifth roots. Hardy was initially skeptical but eventually verified it.

The right-hand side can be simplified:

$$\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} = \sqrt{\frac{5+\sqrt{5}}{2}} - \varphi \tag{57}$$

This theorem exemplifies Ramanujan's extraordinary ability to discover unexpected connections between seemingly unrelated mathematical objects.

10 Derived Identities

10.1 Algebraic Identities from Nested Radicals

From our convergence formula:

$$\sqrt{a+b\sqrt{a+b\sqrt{a+\cdots}}}=\frac{b+\sqrt{b^2+4a}}{2} \tag{58}$$

We can derive:

$$\frac{b+\sqrt{b^2+4a}}{2}=b^{1/2}\cdot(b\cdot b^{1/2}\cdot(b\cdot b^{1/2}\cdot(b\cdot b^{1/2}\ldots)^{1/2})^{1/2})^{1/2} \tag{59}$$

Setting $b=a^2$:

$$\frac{a^2+\sqrt{a^4+4a}}{2}=a\cdot\sqrt{a^2\cdot\sqrt{a^2\cdot\sqrt{a^2\cdot\sqrt{\cdots}}}} \tag{60}$$

10.2 The Identity for $\sqrt{2}$

Setting $b=2$ and $a=1/4$ appropriately:

$$2=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} \tag{61}$$

Wait—this isn't quite right. Let's derive it properly.
For $a=2$:

$$L=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}\implies L^2=2+L\implies L=2 \tag{62}$$

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}=2 \tag{63}$$

This can also be written as:

$$\sqrt{2}=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} \tag{64}$$

More generally:

$$2\cos\left(\frac{\pi}{2^n}\right)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n-1\text{ nested radicals}} \tag{65}$$

As $n\to\infty$, $\cos(\pi/2^n)\to 1$, confirming our limit of 2.

11 Calculus of Nested Radicals

11.1 Functions Defined by Nested Radicals

Define the sequence of functions:

$$f_0(x)=k\quad\text{(constant)} \tag{66}$$

$$f_{n+1}(x)=\sqrt{x+f_n(x)} \tag{67}$$

Explicitly:

$$f_1(x) = \sqrt{x+k} \tag{68}$$

$$f_2(x) = \sqrt{x+\sqrt{x+k}} \tag{69}$$

$$f_3(x) = \sqrt{x+\sqrt{x+\sqrt{x+k}}} \tag{70}$$

$$\vdots \tag{71}$$

$$f_n(x) = \underbrace{\sqrt{x+\sqrt{x+\cdots+\sqrt{x+k}}}}_{n \text{ radicals}} \tag{72}$$

11.2 Differentiability

Theorem 11.1. For $x > 0$ and $k \geq 0$, each $f_n(x)$ is differentiable, and:

$$f_n'(x) = \frac{1}{2f_n(x)}(1 + f_{n-1}'(x)) \tag{73}$$

with $f_0'(x) = 0$.

Proof. By the chain rule:

$$f_{n+1}'(x) = \frac{d}{dx}\sqrt{x+f_n(x)} = \frac{1+f_n'(x)}{2\sqrt{x+f_n(x)}} = \frac{1+f_n'(x)}{2f_{n+1}(x)} \tag{74}$$

□

Computing the first few derivatives:

$$f_1'(x) = \frac{1}{2\sqrt{x+k}} = \frac{1}{2f_1(x)} \tag{75}$$

$$f_2'(x) = \frac{1}{2f_2(x)}\left(1 + \frac{1}{2f_1(x)}\right) \tag{76}$$

$$f_3'(x) = \frac{1}{2f_3(x)}\left(1 + \frac{1}{2f_2(x)}\left(1 + \frac{1}{2f_1(x)}\right)\right) \tag{77}$$

This generates a beautiful nested structure in the derivatives themselves!

General Formula:

$$f_n'(x) = \frac{1}{2f_n}\left(1 + \frac{1}{2f_{n-1}}\left(1 + \frac{1}{2f_{n-2}}\left(1 + \cdots + \frac{1}{2f_1}\right)\right)\right) \tag{78}$$

11.3 Integration

The antiderivatives of $f_n(x)$ become increasingly complex.

Theorem 11.2. $\int f_1(x)\,dx = \frac{2}{3}(x+k)^{3/2} + C$

Proof. Direct integration:

$$\int \sqrt{x+k}\, dx = \frac{2}{3}(x+k)^{3/2} + C \tag{79}$$

□

For $f_2(x) = \sqrt{x + \sqrt{x+k}}$, the integral is considerably more complex. Setting $k = 1$:

$$\int \sqrt{x + \sqrt{x+1}}\, dx = \frac{1}{12}\sqrt{x + \sqrt{x+1}}\left(2(x + \sqrt{x+1}) + 6\sqrt{x+1} + 2x - 11\right) + C' \tag{80}$$

where C' involves logarithmic terms.

For $n \geq 3$, closed-form antiderivatives in terms of elementary functions are generally not available, though the functions remain integrable over any bounded interval.

12 Open Problems and Further Directions

12.1 Algebraic Independence

1. Is $\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \cdots}}}}$ algebraic or transcendental?
2. What is the exact value of $\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \cdots}}}}$?

Numerical computation gives approximately 1.7579327566 …, but no closed form is known.

12.2 Generalized Nested Radicals

1. Study of $\sqrt[n]{a_1 + \sqrt[n]{a_2 + \sqrt[n]{a_3 + \cdots}}}$ for $n > 2$.
2. Mixed-degree nested radicals.
3. Complex-valued nested radicals.

12.3 Connections to Modular Forms

Ramanujan's wild theorem suggests deep connections between nested radicals and modular forms. This remains an active area of research.

13 Conclusion

Ramanujan's work on nested radicals demonstrates his remarkable ability to find patterns and connections in mathematics. From a simple binomial expansion, he constructed an elegant theory relating nested radicals to continued fractions, the golden ratio, and ultimately to modular forms and q-series.

The key results of this monograph include:

1. **Ramanujan's General Formula**: $x + n + a$ can be expressed as an infinite nested radical.
2. **Convergence**: Constant-coefficient nested radicals $\sqrt{a + \sqrt{a + \cdots}}$ converge to $(1+\sqrt{1+4a})/2$.

3. **Duality**: Nested radicals and continued fractions are intimately connected, both yielding the same algebraic expressions.

4. **Special Values**: The golden ratio $\varphi = (1 + \sqrt{5})/2$ arises as the simplest non-trivial nested radical.

5. **Calculus**: Functions defined by nested radicals are differentiable and integrable, with derivatives exhibiting their own nested structure.

The theory of nested radicals continues to yield surprises, connecting elementary algebra to deep areas of number theory and analysis.

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A Numerical Values

Expression	Approximate Value
$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$	1.6180339887 ... (golden ratio)
$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$	2.0000000000 ...
$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots}}}}$	1.7579327566 ...
$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}}$	3.0000000000 ...
$\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}$	1.3247179572 ... (plastic constant)

Table 1: Numerical values of selected nested radicals

B Viète's Formula

François Viète discovered in 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots \tag{81}$$

This can be written as:

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{a_n}{2} \tag{82}$$

where $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$.

Since $a_n = 2\cos(\pi/2^{n+1})$, we have:

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2^{n+1}}\right) \tag{83}$$

This beautiful identity connects nested radicals to π, predating Ramanujan by over 300 years.

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