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Elementary Analysis on Ramanujan's Nested Radicals

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Contents

Contents.....	1
Abstract	2
About Ramanujan.....	3
Introduction to Nested Radicals	4
Definitions	4
Denesting	5
How to denest nested radicals?.....	5
Introducing Ramanujan's Nested Radicals.....	7
Nested radicals of type $a + \sqrt{ba + b}$	13
Convergence of Ramanujan's nested radicals.....	14
Nested Radicals as Continued fractions and <i>vice-versa</i>	15
Convergence of $n - th$ degree nested radicals	17
General Approach to the convergence of Nested Radicals.....	18
Derivated identities from nested radicals.....	18
Ramanujan's Ultimate nested radicals	21
Ramanujan's wild theorem on nested radicals.....	22
Calculus of Nested Radicals.....	22
Differentiability.....	23
Integration.....	24
References.....	25
Tools used.....	25
Regards	25

Abstract

Indian mathematical prodigy Srinivasa Ramanujan needs no introduction due to his remarkable contributions to various fields of mathematics. In his brief lifetime, he discovered and published several theorems and formulas which even today surprize the leading math-majors. This project is based on one of his very first contributions to mathematics, in forms of some excellently patterned and formulized *nested radicals*.

This is a summary of the proper research about Ramanujan's Nested Radicals, initiated first at MY DIGITAL NOTEBOOK¹ and done rigorously in recent months. We shall start with basic definition of nested radicals and will end with the analytic properties. In between we'll explore how good Ramanujan's work on nested radicals go.

On the elementary basis, the theory of *Ramanujan's radicals* is almost complete in reference and has hardly any chance to find a research topic in it. My aim is, however to build strong basics in the topic at the instance and to do proper research in upcoming days by learning more about *algebraic*, *analytic* and *modular* aspects of number theory.

¹ <http://gauravtiwari.org/nested/> & <http://gauravtiwari.org/ramanujan/>

Brief about Ramanujan

Name:	Srinivasa Iyengar Ramanujan
Born:	December 22 nd 1887. Erode, Madras, Tamil Nadu.
Lived:	At Kumbakonam, Tamil Nadu
Studied:	Government Arts College Pachaiyappa's College
Died:	April 26 th 1920 (32 years old). Chetput, Madras, Tamil Nadu
Well-known for:	Ramanujan's Notebooks Ramanujan's Nested Radicals Mock Theta function and Ramanujan Theta function Landau-Ramanujan Constant Ramanujan conjecture Ramanujan Prime Ramanujan-Soldner Constant Ramanujan's Sum Roger-Ramanujan Identities Master Theorem Some Properties of Bernoulli's Numbers

Introduction to Nested Radicals

Definitions

A nested radical is a number of following type:

$$\sqrt{x_1 + x_2 \sqrt{x_3 + x_4 \sqrt{x_5 + x_6 \sqrt{x_7 + x_8 \sqrt{\dots}}}}} \quad \text{where } x_i \in \mathbf{R} \quad \dots (1)$$

Nested radicals, after huge contributions of Ramanujan to it, are also called Ramanujan's Radicals.

A simple example of nested radical is

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\dots}}}}}$$

The radicals may have not only 'square roots' nested but can also be complicated having 'cube roots', ... or 'n-th roots'.

$\sqrt[4]{5 + \sqrt[3]{11 + \sqrt{23}}}$ is also an example of a 'nested' radical. In this project, our main aim will be on nested radicals with square-roots only, unless stated otherwise.

The nested radicals may repeat finitely or infinitely and so they have their names of *finite nested radicals* and *infinite nested radicals*. $\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4}}}}$ is

a finite nested radical with *four-nests* while $\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\dots}}}}}$ is an infinite nested radical.

Denesting

The process of reducing the radicals from a number is called **denesting**. Denesting is the first step toward finding a value of a number with nested radical.

For example: $\sqrt{3 + 2\sqrt{2}}$ can be easily written as: $1 + \sqrt{2}$, or in other words, can be denested.

How to denest nested radicals?

DENESTING NESTED RADICALS WITH 2-NESTS

Denesting finite nested radicals with two-nests is simple and can be easily done by applying algebraic methods. Assume $\sqrt{a + b\sqrt{c}}$ be a number with two nests and we are going to convert it into more systematic number like $\sqrt{d} + \sqrt{e}$ with only one nest.

Let

$$\sqrt{a + b\sqrt{c}} = \sqrt{d} + \sqrt{e}$$

Squaring both sides,

$$a + b\sqrt{c} = d + e + 2\sqrt{de}$$

Equating integer and radical parts to each other

$$a = d + e \dots (2)$$

$$b\sqrt{c} = 2\sqrt{de} \dots (3)$$

Solving equation (3) by squaring both sides again

$$b^2c = 4de \dots (4)$$

If we put given values of a , b and c , we can get the solution for d and e . From (2)

$$d = a - e \dots (5)$$

Putting this in (4), we get

$$b^2c = 4(a - e)e$$

Or,

$$4e^2 - 4ae + b^2c = 0 \dots (6)$$

Solving equation (6) yields two solutions for e and thus two solutions for d . Thus, there may be two ways to write a nested radical with 2-nests. But, while writing the solution, always ignore the value of e which is 'not real' for real-valued nested radical.

DENESTING NESTED RADICALS WITH 3-NESTS

This time, consider a nested radical of type $\sqrt{a + b\sqrt{c + d\sqrt{e}}}$. Equating it to $\sqrt{f} + \sqrt{g + \sqrt{h}}$, we have

$$\sqrt{a + b\sqrt{c + d\sqrt{e}}} = \sqrt{f} + \sqrt{g + \sqrt{h}}$$

Squaring both sides,

$$a + b\sqrt{c + d\sqrt{e}} = f + g + \sqrt{h} + 2\sqrt{fg + f\sqrt{h}}$$

Now, as

$$a = f + g \dots (7)$$

And,

$$b\sqrt{c + d\sqrt{e}} = \sqrt{h} + 2\sqrt{fg + f\sqrt{h}} \dots (8)$$

On squaring both sides again,

$$b^2c + b^2d\sqrt{e} = h + 4fg + 4f\sqrt{h} + 4\sqrt{fgh + fh\sqrt{h}}$$

This gives,

$$b^2c = h + 4fg \dots (9)$$

And,

$$b^2d\sqrt{e} = 4f\sqrt{h} + 4\sqrt{fgh + fh\sqrt{h}}$$

$$b^4d^2e = 16f^2h + 16fgh + 16fh^{\frac{3}{2}} + 32f\sqrt{fgh^2 + fh^{\frac{5}{2}}} \dots (10)$$

Solving, equation (7), (9) and (10) will give the required values of f , g and h . Using them the radical gets denested to one level. Further denesting is even more complicated, so I'll stop here.

Susan Landau introduced algorithms to denest such finite nested radicals, which is a topic out of the discussion here. There is a paper by *J. Blommer*, Berlin University, in which he has discussed the algebraic aspects of Nested Radicals by working on Landau's algorithm.

Introducing Ramanujan's Nested Radicals

In algebra, the simplest binomial theorem of degree 2 can be written as

$$(x + a)^2 = x^2 + 2xa + a^2 \dots (11)$$

Replacing a by $n+a$ in the above equation, where $x, n, a \in \mathbf{R}$.

Therefore

$$(x + (n + a))^2 = x^2 + 2x(n + a) + (n + a)^2$$

$$\begin{aligned}
(x+n+a)^2 &= x^2 + 2xn + 2xa + (n+a)^2 \\
&= ax + (n+a)^2 + x^2 + 2xn + ax \\
&= ax + (n+a)^2 + x(x+2n+a) \dots (13)
\end{aligned}$$

Taking the positive² square-roots of both the sides

$$x+n+a = \sqrt{ax + (n+a)^2 + x(x+2n+a)} \dots (14)$$

In equation (14) if we replace x by $x+n$, we obtain

$$\begin{aligned}
(x+n)+n+a &= \sqrt{a(x+n) + (n+a)^2 + (x+n)((x+n)+2n+a)} \\
x+2n+a &= \sqrt{a(x+n) + (n+a)^2 + (x+n)(x+3n+a)} \dots (15)
\end{aligned}$$

Similarly,

$$x+3n+a = \sqrt{a(x+2n) + (n+a)^2 + (x+2n)(x+4n+a)} \dots (16)$$

And,

$$x+4n+a = \sqrt{a(x+3n) + (n+a)^2 + (x+3n)(x+5n+a)} \dots (17)$$

Inducing these results to (larger) $k \in \mathbf{N}$ terms³, we have

$$x+kn+a = \sqrt{a(x+(k-1)n) + (n+a)^2 + (x+(k-1)n)(x+(k+1)n+a)} \dots (18)$$

Now, putting the value of $x+2n+a$ from equation (15) to equation (14), we get

$$x+n+a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)(x+3n+a)}} \dots (19)$$

² As x, n, a all are real numbers, we can adjust the sign of square roots according to them.

³ k may be as big but finite as possible and may also tend to ∞ .

Again, putting the value of $x+3n+a$ from equation (16) to equation (14), we get

$$x + n + a = \sqrt{ax + (n + a)^2 + x \sqrt{a(x + n) + (n + a)^2 + (x + n) \sqrt{a(x + n) + (n + a)^2 + (x + 2n)(x + 4n + a)}} \dots (20)$$

Generalizing these sets of equations to k –terms we obtain k-nested radicals

$$x + n + a = \sqrt{ax + (n + a)^2 + x \sqrt{a(x + n) + (n + a)^2 + (x + n) \sqrt{a(x + 2n) + (n + a)^2 + (x + 2n) \sqrt{\dots + (x + (k - 2)n) \sqrt{a(x + (k - 1)n) + (n + a)^2 + x(x + (k + 1)n + a)}}}} \dots (21)$$

This is the general formula of Ramanujan's nested radicals derived from a binomial expression. ■

Now, we shall discuss and derive the results and special cases, we obtain from above equations ranging from equation (14) to equation (21).

- Interchanging⁴ the algebraic symbols x , n and a , we get

$$x + n + a = \sqrt{ax + (n + a)^2 + x \sqrt{a(x + n) + (n + a)^2 + (x + n) \sqrt{a(x + 2n) + (n + a)^2 + (x + 2n) \sqrt{\dots + (x + (k - 2)n) \sqrt{a(x + (k - 1)n) + (n + a)^2 + x(x + (k + 1)n + a)}}}} \dots$$

⁴ As x , n and a all are real numbers, they can be interchanged by each other's values.

$$\begin{aligned}
&= \sqrt{nx + (n+a)^2 + x} \sqrt{n(x+a) + (n+a)^2 + (x+a)} \sqrt{n(x+2a) + (n+a)^2 + (x+2a)} \sqrt{\frac{\dots + (x+(k-2)a)}{\sqrt{n(x+(k-1)a) + (n+a)^2 + x(x+(k+1)a+n)}}} \\
&= \sqrt{an + (x+a)^2 + n} \sqrt{a(x+n) + (x+a)^2 + (x+n)} \sqrt{a(n+2x) + (x+a)^2 + (n+2x)} \sqrt{\frac{\dots + (n+(k-2)x)}{\sqrt{a(n+(k-1)x) + (x+a)^2 + n(n+(k+1)x+a)}}}
\end{aligned}$$

- Putting $n=0=a$ and $x \neq 0$ in equation (21) We get,

$$x = \sqrt{x \sqrt{x \sqrt{x \sqrt{\dots + x \sqrt{x^2}}}}}$$

It is a very cool and easily solvable nested radical, which alternatively equals to $x = \sqrt{x \sqrt{x^2}} = \sqrt{x \cdot x}$.

Taking k as large as possible⁵, we get

$$x = \sqrt{x \sqrt{x \sqrt{\dots}}}$$

- Putting $x=l=n=a$

⁵ i.e., $k \rightarrow \infty$

$$1 + 1 + 1$$

$$= \sqrt{1 + (1 + 1)^2 + 1} \sqrt{1(1 + 1) + (1 + 1)^2 + (1 + 1)} \sqrt{1(1 + 2 \cdot 1) + (1 + 1)^2 + (1 + 2 \cdot 1)} \sqrt{\dots + (1 + (k - 2)1) + \sqrt{1(1 + (k - 1)1) + (1 + 1)^2 + 1(x + (k + 1)1 + 1)}}$$

$$3 = \sqrt{5 + 1} \sqrt{6 + 2} \sqrt{7 + 3} \sqrt{8 + 4} \sqrt{\dots + (k - 1)\sqrt{2k + 7}}$$

For larger values of k we have,

$$3 = \sqrt{5 + 1} \sqrt{6 + 2} \sqrt{7 + 3} \sqrt{8 + 4\sqrt{\dots}}$$

- 3 can also be obtained by putting $x=2$ in adjusted form of Ramanujan's general formula of nested radicals, which is

$$x + 1 = \sqrt{1 + x} \sqrt{1 + (x + 1)} \sqrt{1 + (x + 2)} \sqrt{1 + (x + 3)} \sqrt{\dots}$$

- Also, $n = -1$ and $a = 1$ in equation (21), we get

$$x = \sqrt{x+x} \sqrt{(x-1)+(x-1)} \sqrt{(x-2)+(x-2)} \sqrt{\dots + (x-(k-2)n)} \sqrt{\sqrt{(x-(k-1)n) + x(x-(k+1)n+1)}}$$

The same after replacing x by $1+x$ becomes,

$$1+x = \sqrt{1+x+(1+x)} \sqrt{x+x} \sqrt{(x-1)+(x-1)} \sqrt{\dots + (1+x-(k-2)n)} \sqrt{\sqrt{(1+x-(k-1)n) + (1+x)(x-(k+1)n+2)}}$$

- Putting $x=a=n(=n)$ we get

$$3n = \sqrt{n^2 + 4n^2 + n} \sqrt{2n^2 + 4n^2 + 2n} \sqrt{3n^2 + 4n^2 + 3n} \sqrt{\dots + ((k-1)n)} \sqrt{\sqrt{(n^2 + (k-1)n^2) + 4n^2 + ((k+2)n^2 + n)}}$$

$$3n = \sqrt{5n^2 + n} \sqrt{6n^2 + 2n} \sqrt{7n^2 + 3n} \sqrt{\dots + ((k-1)n)} \sqrt{\sqrt{(2k+6)n^2 + n}}$$

Eventually, we have derived an expression for nested radicals of following type:

$$z = \sqrt{a + b\sqrt{c + d\sqrt{e + \dots}}} \dots (22)$$

Next to this, we'll look into the special properties of the different cases of Ramanujan's nested radicals of type (22)

Nested radicals of type $\sqrt{a + b\sqrt{a + b\sqrt{\dots}}}$

Clearly, $\sqrt{1 + 2\sqrt{3 + \sqrt{4 + \sqrt{5 + \sqrt{\dots}}}}}$ is an example of such a nested radical. In other examples, some important ones can be obtained by putting $a = b = 1$ which yields

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\dots}}}}$$

It can be solved easily by assuming

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\dots}}}} = X$$

Squaring both sides,

$$1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\dots}}}} = X^2 \dots (23)$$

Now as, $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\dots}}}} = X$, we can write (23) as

$$1 + X = X^2$$

Which on solution gives⁶

$$X = \frac{\sqrt{5} + 1}{2}$$

the golden ratio ϕ . Thus,

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\dots}}}} = \phi = \frac{\sqrt{5} + 1}{2} \dots (24)$$

Convergence of Ramanujan's nested radicals⁷

Let the nested radical of type $\sqrt{a + b\sqrt{a + b\sqrt{\dots}}}$ converges to a limit ℓ for $a \geq 0$ and $b \geq 0$. Then

⁶ As usual there will be two solutions, one $X = \frac{\sqrt{5}+1}{2}$ and other $X = \frac{-\sqrt{5}+1}{2}$, out of which only 1st one is acceptable as X must always be positive regarding the provided radical.

⁷ We will ignore negative solutions now on.

$$\begin{aligned} \ell &:= \sqrt{a + b\sqrt{a + b\sqrt{\dots}}} \\ \ell &:= \sqrt{a + b\ell} \\ \ell^2 - b\ell - a &:= 0 \\ \Leftrightarrow \ell &= \frac{b + \sqrt{b^2 + 4a}}{2} \dots (25) \end{aligned}$$

Hence, $\sqrt{a + b\sqrt{a + b\sqrt{\dots}}}$ always converges to $\frac{b + \sqrt{b^2 + 4a}}{2}$, $\forall a, b \in \mathbb{Z}^+ \cup \{0\}$.

To verify, we observe that when

- $a=0$.

$$\sqrt{b\sqrt{b\sqrt{\dots}}} = \frac{b + \sqrt{b^2}}{2} = \frac{2b}{2} = b.$$

- $b=0$. The expression eventually reduces to \sqrt{a} .

$$\sqrt{a + 0\sqrt{\text{some number}}} = \sqrt{a}$$

Nested Radicals as Continued fractions and *vice-versa*

Assume an infinite continued fraction of type

$$a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \dots}}}$$

converges when $a \geq 0$ and $b \geq 0$, and let ℓ be the limit. Then

$$\begin{aligned}\ell &= a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \dots}}} \\ \ell &= a + \frac{b}{\ell} \\ \ell^2 - a\ell - b &= 0 \\ \Leftrightarrow \ell &= \frac{a + \sqrt{a^2 + 4b}}{2} \dots (26)\end{aligned}$$

Comparing equations (25) and (26), we find that if we interchange a and b , in equation (26),

$$b + \frac{a}{b + \frac{a}{b + \frac{a}{b + \dots}}} = \frac{b + \sqrt{b^2 + 4a}}{2} \dots (27)$$

As, R.H.S. of both (25) and (27) are equal, LHS will be equal too.

Therefore,

$$\sqrt{a + b\sqrt{a + b\sqrt{\dots}}} = b + \frac{a}{b + \frac{a}{b + \frac{a}{b + \dots}}} = \frac{b + \sqrt{b^2 + 4a}}{2} \dots (28)$$

This is one of the most important results for nested radical's convergence.

Substituting $a = 1 = b$ in equation (28), we have

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = \frac{1 + \sqrt{5}}{2}$$

and these equal to the golden ratio ϕ as quoted at equation (24).

Convergence of $n - th$ degree nested radicals

Let the nested radical of type $\sqrt[n]{a + b\sqrt[n]{a + b\sqrt[n]{a + \dots}}}$ converges to a limit ℓ for $a \geq 0$ and $b \geq 0$. Then

$$\ell := \sqrt[n]{a + b\sqrt[n]{a + b\sqrt[n]{a + \dots}}} \dots (29)$$

$$\ell := \sqrt[n]{a + b\ell}$$

$$\Rightarrow \ell^n - b\ell - a = 0 \dots (30)$$

To solve it, let $a = \alpha\ell^n$ and $b = \beta\ell^{n-1}$, then

$$\ell^n - \beta\ell^n - \alpha\ell^n = 0$$

$$1 - \beta - \alpha = 0$$

$$\beta = 1 - \alpha \dots (31)$$

Substituting the values of α & β to equation (30), we get

$$\ell := \sqrt[n]{\alpha\ell^n + \beta\ell^{n-1} \sqrt[n]{\alpha\ell^n + \beta\ell^{n-1} \sqrt[n]{\alpha\ell^n + \dots}}}$$

that is, the point to which the given n -nest radical converges.

General Approach to the convergence of Nested Radicals

Consider a Ramanujan's Radical of type $\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \sqrt{\dots + \sqrt{a_n + \sqrt{\dots}}}}}}$, which elements form a sequence with n th term a_n . For the given number, the n th term of the sequence of radicals is $\left((a_n)^{\frac{1}{2^n}}\right)$. Now as according to Herschfeld's theorem, a sequence $\langle f_n \rangle$ converges if and

only if $f_n^{2^{-n}}$ converges, $\forall f_i \geq 0$. It's clear explanation, $\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \sqrt{\dots + \sqrt{a_n + \sqrt{\dots}}}}}}$ converges in \mathbf{R} as $a_i^{2^{-n}}$ converges for all $a_i \geq 0$.

Derivated identities from nested radicals

As we proved,

$$\sqrt{a + b\sqrt{a + b\sqrt{a + \dots}}} = \frac{b + \sqrt{b^2 + 4a}}{2}$$

Writing it in other way,

$$\frac{b + \sqrt{b^2 + 4a}}{2} = \sqrt{a + b\sqrt{a + b\sqrt{a + b\sqrt{a + \dots}}}}$$

$$\begin{aligned}
&= \sqrt{a + \sqrt{ab^2 + b^3} \sqrt{a + b\sqrt{a} + \dots}} \\
&= \sqrt{a + \sqrt{ab^2 + \sqrt{ab^6 + b^7} \sqrt{a + \dots}}} \\
&= \sqrt{a + \sqrt{ab^2 + \sqrt{ab^6 + \sqrt{ab^{14} + \dots}}} \\
&= \sqrt{\frac{a}{b^2} \cdot b^2 + \sqrt{\frac{a}{b^2} \cdot b^4 + \sqrt{\frac{a}{b^2} \cdot b^8 + \sqrt{\frac{a}{b^2} \cdot b^{16} + \dots}}}
\end{aligned}$$

Setting $\omega = \frac{a}{b^2}$ or $a = \omega b^2$, we obtain,

$$\begin{aligned}
\frac{b + \sqrt{b^2 + 4a}}{2} &= \sqrt{\frac{a}{b^2} \cdot b^2 + \sqrt{\frac{a}{b^2} \cdot b^4 + \sqrt{\frac{a}{b^2} \cdot b^8 + \sqrt{\frac{a}{b^2} \cdot b^{16} + \dots}}} \\
\frac{b + \sqrt{b^2 + 4\omega b^2}}{2} &= \sqrt{\omega \cdot b^2 + \sqrt{\omega \cdot b^4 + \sqrt{\omega \cdot b^8 + \sqrt{\omega \cdot b^{16} + \dots}}}
\end{aligned}$$

$$\frac{b + b\sqrt{1 + 4\omega}}{2} = \sqrt{\omega \cdot b^2 + \sqrt{\omega \cdot b^4 + \sqrt{\omega \cdot b^8 + \sqrt{\omega \cdot b^{16} + \dots}}} \dots (32)$$

Equation (32) is a very useful algebraic expression, which gives rise to several algebraic numbers viz. rational and irrational.

Putting $\omega = 2$ and $\beta = \frac{1}{2}$ in (32)

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4 \cdot 2}}{2} = \sqrt{2 \cdot \frac{1}{4} + \sqrt{2 \cdot \frac{1}{16} + \sqrt{2 \cdot \frac{1}{256} + \sqrt{2 \cdot \frac{1}{65536} + \dots}}}$$

$$1 = \sqrt{2^{-1} + \sqrt{2^{-3} + \sqrt{2^{-7} + \sqrt{2^{-15} + \dots}}} \dots (33)$$

Multiplying both sides by $\sqrt{2}$:

$$\sqrt{2} = \sqrt{2^0 + \sqrt{2^{-1} + \sqrt{2^{-3} + \sqrt{2^{-7} + \dots}}} \dots (34)$$

Further, the generalization is

$$2^{1-2^k} = \sqrt{2^{1-2^{k+1}} + \sqrt{2^{1-2^{k+2}} + \sqrt{2^{1-2^{k+3}} + \dots}}} \dots (35)$$

Ramanujan's Ultimate nested radicals

As we derived an expression for $(x+n+a)$ in equation (21), we shall attempt to derive the ultimate expressions Ramanujan gave for summation of any two numbers $(x+a)$ by creating another telescopic pattern of nested radicals.

Starting with the same basic idea we used earlier,

$$(x + a)^2 = x^2 + a^2 + 2ax$$

Taking square-root and arranging in a way that

$$\begin{aligned} x + a &= \sqrt{a^2 + x^2 + ax + ax} \\ x + a &= \sqrt{a^2 + x(x + a + a)} \dots (36) \end{aligned}$$

Replacing x by $x+a$ in (36),

$$x + a + a = \sqrt{a^2 + (x + a)(x + 2a + a)} \dots (37)$$

Similarly,

$$x + 2a + a = \sqrt{a^2 + (x + 2a)(x + 3a + a)} \dots (38)$$

By putting the respective values to (36), we derive

$$\begin{aligned} x + a &= \sqrt{a^2 + x(x + a + a)} \\ \Rightarrow x + a &= \sqrt{a^2 + x\sqrt{a^2 + (x + a)(x + 2a + a)}} \end{aligned}$$

Generalizing upto n -terms,

$$x + a = \sqrt{a^2 + x \sqrt{a^2 + (x + a) \sqrt{a^2 + (x + 2a) \sqrt{\dots \sqrt{a^2 + (x + a)(x + (n + 1)a) \dots}}} \dots} \quad (39)$$

This too, give rise to various possibilities under the game of number-theory.

Ramanujan's wild theorem on nested radicals

The letter Ramanujan posted to G.H. Hardy on 16th January 1913, had the following *wild theorem*⁸ in it:

$$\frac{1}{1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \dots}}}} = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2} \right) \sqrt[5]{e^{2\pi}} \quad \dots(40)$$

The wild theorem, then taken as surprize by Hardy, is a relevant example of how an infinite continued fraction can be equal to a finite nested radical.

Calculus of Nested Radicals

A Ramanujan number can be considered to be formed by a recurrence relation, e.g.; $f_{n+1}(x) := \sqrt{x + f_n(x)}$ with $f_0(x) = k$ (constant).

⁸ Source: Paramanad's Math Notes: <http://paramanands.blogspot.in/2013/09/>

Therefore, $f_1(x) = \sqrt{x + f_0(x)} = \sqrt{x + k}$

Similarly, $f_2(x) = \sqrt{x + \sqrt{x + k}}$

and , $f_2(x) = \sqrt{x + \sqrt{x + \sqrt{x + k}}}$

.....

$$f_n(x) = \sqrt{x + \sqrt{x + \sqrt{\dots + \sqrt{x + k} \dots}}} \quad (41)$$

Where $f_n(x)$ is a Ramanujan radical with n - x 's and n -radicals.

Differentiability

$f_1(x)$ is easily differentiable and has the derivative, $f_1'(x) = \frac{1}{2} \frac{1}{\sqrt{x+k}} = \frac{1}{2} \left(\frac{1}{f_1}\right)$. The next, $f_2(x)$ can also be differentiated as,

$$\frac{df_2}{dx} = \frac{1}{2f_2} \left(1 + \frac{1}{2f_1}\right) = \frac{1}{f_2} \left(\frac{1 + f_1'}{2}\right) \dots (42)$$

Similarly, $f_3(x)$ has the differential,

$$\frac{df_3}{dx} = \frac{1}{2f_3} \left(1 + \frac{1 + \frac{1}{2f_1}}{2f_2}\right) = \frac{1}{f_3} \left(\frac{1 + f_2'}{2}\right) \dots (43)$$

Generalizing it up-to n terms,

$$\frac{df_n}{dx} := \frac{1}{2f_{n-1}} \left(1 + \frac{1}{2f_{n-2}} \left(1 + \frac{1}{2f_{n-3}} \left(1 + \frac{1}{2f_{n-4}} \left(1 + \dots + \frac{1}{2f_3} \left(1 + \frac{1}{2f_2} \left(1 + \frac{1}{2f_1}\right)\right)\right)\right)\right)\right) \dots (44)$$

Also,

$$\frac{df_n}{dx} = \frac{1}{2f_{n-1}} (1 + f'_{n-2}) \dots \quad (45)$$

Equation (44) and (45) clearly elaborates that the function $f_n(x)$ defined by (41) is differentiable and has a beautiful recurrence relation.

Integration

From (45), it's highly probable that $f_n; n \in \mathbf{N}$ is an integrable function in \mathbf{R} . But $\int f_n$ has equally less chance of being represented in terms of elementary functions $\forall n \geq 3$. This can be understood by the following illustration:

$$\int f_1 := \frac{2}{3}(\sqrt{x+k})^3 + c_1 \text{ where } c_1 \text{ is an integration constant.}$$

If we ignore the integrating constants, then; $\int f_1 := \frac{2}{3}f_1^3$.

Similarly,

$$\int f_2 := \frac{1}{12}f_2 \cdot (2f_2^2 + 6f_1^2 + 2k - 11) + 15 \log(2f_2 + 2f_1 + 1) + c_2$$

Putting $k=1$, we obtain,

$$\int f_2 := \left(\frac{1}{6}f_2^3 + \frac{1}{2}f_1^2f_2 - \frac{3}{4}f_2 \right) + 15 \log(2f_2 + 2f_1 + 1) \quad : \text{ ignoring } c_2$$

And,

$$\int f_3 = \int \frac{4((f_1+f_2)^4 + (f_1+f_2)^3 + (f_1+f_2-1))}{(2f_1+2f_2+1)^3} \sqrt{((f_1+f_2)^2+1)^2 + (f_1+f_2-1)(2f_1+2f_2+1)^2}$$

Which is highly unlikely to have a chance to be expressed in the terms of elementary functions. However, f_3 is integrable in \mathbf{R} just like other f_i are.

References

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2. Elementary Number Theory, D M Burton
3. Ramanujan's Notebooks, Rao, imsc.res.in
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Tools used

Following tools were used to produce the best print of text and math graphics:

- Microsoft Word 2013 : Used for publishing content and applying simple formulae.
- MiKTeX and TeX-studio for LaTeX typesetting and complicated formulae.

Regards

Due to large formula, I decided to print the complete project in landscape mode. The readers are advised to bear the trouble. A printable PDF copy of this document can be accessed at gauravtiwari.org/~projects/. Further updates to this project will be released on the same page.