Solving Integral Equations
The Interactive Way!

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Contents

Chapter 1

Basics

1.1 What is an Integral Equation?

An integral equation is an equation in which an unknown function appears under one or more integration signs. Any integral calculus statement like – $y = \int_a^b \phi(x)dx$ or $y(x) = \int_a^x \psi(x)dx$ can be considered as an integral equation. If you noticed I have used two types of integration limits in above integral equations –their significance will be discussed later in the book. A general type of integral equation, $g(x)y(x) = f(x) + \lambda \int_a^\square K(x,t)y(t)dt$ is called linear integral equation as only linear operations are performed in the equation. The one, which is not linear, is obviously called 'Non-linear integral equation'. In this book, when you read 'integral equation' understand it as 'linear integral equation' until specified.

In the general type of the linear equation $g(x)y(x) = f(x)+\lambda \int_a^\square K(x,t)y(t)dt$ we have used a 'box $\square$' to indicate the higher limit of the integration. Integral Equations can be of two types according to whether the box $\square$ (the upper limit) is a constant (b) or a variable (x). First type of integral equations which involve constants as both the limits — are called Fredholm Type Integral equations. On the other hand, when one of the limits is a variable (x, the independent variable of which y, f and K are functions) , the integral equation is called Volterra's Integral Equations. Thus $g(x)y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$ is a **Fredholm Integral Equation** and $g(x)y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt$ is a **Volterra Integral Equation**. In an integral equation, y is to be determined with g, f and K being known and λ being a non-zero complex parameter. The function $K(x,t)$ is called the '**kernel**' of the integral equation.

1.2 Structure of an Integral Equation

1.3 Types of Fredholm Integral Equations

As the general form of Fredholm Integral Equation is $g(x)y(x) = f(x)+\lambda \int_a^b K(x,t)y(t)dt$, there may be following other types of it according to the values of g and f:

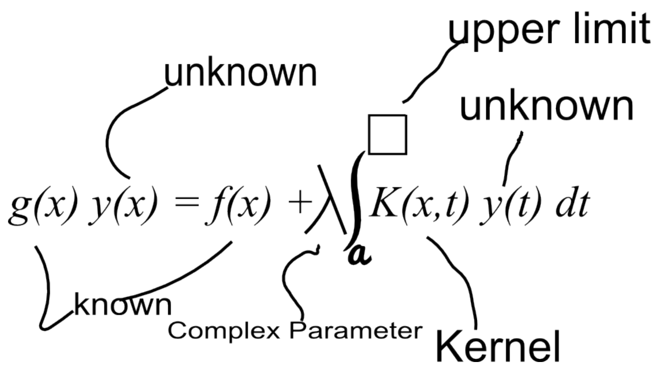

Figure 1.1:

1.3.1 Fredholm Integral Equation of First Kind

When $g(x) = 0$ $f(x) + \lambda \int_a^b K(x,t)y(t)dt = 0$

1.3.2 Fredholm Integral Equation of Second Kind

When $g(x) = 1$ $y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$

1.3.3 Fredholm Integral Equation of Homogeneous Second Kind

When $f(x) = 0$ and $g(x) = 1$ $y(x) = \lambda \int_a^b K(x,t)y(t)dt$

1.3.4 Fredholm Equation of Third Kind

The general equation of Fredholm equation is also called Fredholm Equation of Third/Final kind, with $f(x) \neq 0, 1 \neq g(x) \neq 0$.

1.4 Types of Volterra Integral Equations

As the general form of Volterra Integral Equation is $g(x)y(x) = f(x)+\lambda \int_a^x K(x,t)y(t)dt$, there may be following other types of it according to the values of g and f:

1.4.1 Volterra Integral Equation of First Kind

When $g(x) = 0$ $f(x) + \lambda \int_a^x K(x,t)y(t)dt = 0$

1.4.2 Volterra Integral Equation of Second Kind

When $g(x) = 1$ $y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt$

1.4.3 Volterra Integral Equation of Homogeneous Second Kind

When $f(x) = 0$ and $g(x) = 1$ $y(x) = \lambda \int_a^x K(x,t)y(t)dt$

1.4.4 Volterra Integral Equation of Third Kind

The general equation of Volterra equation is also called Volterra Equation of Third/Final kind, with $f(x) \neq 0, 1 \neq g(x) \neq 0$.

1.5 Singular Integral equations

In the general Fredholm/Volterra Integral equations, there arise two singular situations: • the limit $a \to -\infty$ and $\square \to \infty$. • the kernel $K(x,t) = \pm\infty$ at some points in the integration limit $[a, \square]$. then such integral equations are called Singular Linear Integral Equations. **Type-1:** $a \to -\infty$ and $\square \to \infty$ **General Form:** $g(x)y(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x,t)y(t)dt$ **Example:** $y(x) = 3x^2 + \lambda \int_{-\infty}^{\infty} e^{-|x-t|}y(t)dt$

Type-2: $K(x,t) = \pm\infty$ at some points in the integration limit $[a, \square]$ **Example:** $y(x) = f(x) + \int_0^x \dfrac{1}{(x-t)^n} y(t)$ is a singular integral equation as the integrand reaches to ∞ at $t = x$.

1.6 Kernel

The nature of solution of integral equations solely depends on the nature of the Kernel of the integral equation. Kernels are of following special types:

1.6.1 Symmetric Kernel

When the kernel $K(x,t)$ is symmetric or complex symmetric or Hermitian, if

$$K(x,t) = \bar{K}(t,x)$$

where bar $\bar{K}(t,x)$ denotes the complex conjugate of $K(t,x)$. That's if there is no imaginary part of the kernel then $K(x,t) = K(t,x)$ implies that K is a symmetric kernel. *For example* $K(x,t) = \sin(x+t)$ is symmetric kernel.

1.6.2 Separable or Degenerate Kernel

A kernel $K(x,t)$ is called separable if it can be expressed as the sum of a finite number of terms, each of which is the product of 'a function' of x only and 'a function' of t only, i.e.,

$$K(x,t) = \sum_{n=1}^{\infty} \phi_i(x)\psi_i(t)$$

.

1.6.3 Difference Kernel

When $K(x,t) = K(x-t)$ then the kernel is called difference kernel.

1.6.4 Resolvent or Reciprocal Kernel

The solution of the integral equation $y(x) = f(x) + \lambda \int_a^{\square} K(x,t)y(t)dt$ is of the form $y(x) = f(x) + \lambda \int_a^{\square} \mathfrak{R}(x,t;\lambda)f(t)dt$. The kernel $\mathfrak{R}(x,t;\lambda)$ of the solution is called resolvent or reciprocal kernel.

1.7 Integral Equations of Convolution Type

The integral equation $g(x)y(x) = f(x) + \lambda \int_a^{\square} K(x,t)y(t)dt$ is called of convolution type when the kernel $K(x,t)$ is difference kernel, i.e., $K(x,t) = K(x-t)$. Let $y_1(x)$ and $y_2(x)$ be two continuous functions defined for $x \in E \subseteq \mathbb{R}$ then the convolution of y_1 and y_2 is given by

$$y_1 * y_2 = \int_E y_1(x-t)y_2(t)dt = \int_E y_2(x-t)y_1(t)dt$$

. For standard convolution, the limits are $-\infty$ and ∞.

1.8 Eigenvalues and Eigenfunctions of the Integral Equations

The homogeneous integral equations $y(x) = \lambda \int_a^{\square} K(x,t)y(t)dt$ have the obvious solution $y(x) = 0$ which is called the zero solution or the trivial solution of the integral equation. Except this, the values of λ for which the integral equation has non-zero solution $y(x) \neq 0$, are called the eigenvalues of integral equation or eigenvalues of the kernel. Every non-zero solution $y(x) \neq 0$ is called an eigenfunction corresponding to the obtained eigenvalue λ.

- Note that $\lambda \neq 0$
- If $y(x)$ an eigenfunction corresponding to eigenvalue λ then $c \cdot y(x)$ is also an eigenfunction corresponding to λ.

1.9 Leibnitz Rule of Differentiation under integral sign

Let $F(x,t)$ and $\dfrac{\partial F}{\partial x}$ be continuous functions of both x and t and let the first derivatives of $G(x)$ and $H(x)$ are also continuous, then $\dfrac{d}{dx}\displaystyle\int_{G(x)}^{H(x)} F(x,t)dt = \displaystyle\int_{G(x)}^{H(x)} \frac{\partial F}{\partial x} dt + F(x, H(x))\frac{dH}{dx} - F(x, G(x))\frac{dG}{dx}$. This formula is called Leibnitz's Rule of differentiation under integration sign. In a special case, when G(x) and H(x) both are absolute (constants) –let $G(x) = a$, $H(x) = b \iff dG/dx = 0 = dH/dx$ –then $\dfrac{d}{dx}\displaystyle\int_a^b F(x,t)dt = \int_a^b \frac{\partial F}{\partial x} dt$.

1.10 The Magical Formula

Changing Integral Equation with Multiple integrals into standard simple integral.

The integral of order n is given by $\displaystyle\int_{\Delta}^{\square} f(x)dx^n$. We can prove that $\displaystyle\int_a^t f(x)dx^n = \int_a^t \frac{(t-x)^{n-1}}{(n-1)!} f(x)dx$

Example: Solve $\int_0^1 x^2 dx^2$.

Solution: $\int_0^1 x^2 dx^2$

$= \int_0^1 \frac{(1-x)^{2-1}}{(2-1)!} x^2 dx$ (since t=1)

$= \int_0^1 (1-x) x^2 dx$

$= \int_0^1 (1-x) x^2 dx$

$= \int_0^1 (x^2 - x^3) dx = 1/12$ □

Chapter 2

$\mathfrak{L}_2$ function

2.1 Square Integrable function or quadratically integrable function $\mathfrak{L}_2$ function

A function $y(x)$ is said to be square integrable or $\mathfrak{L}_2$ function on the interval (a, b) if

$$\int_a^b |y(x)|^2 dx < \infty$$

or

$$\int_a^b y(x)\bar{y}(x) dx < \infty$$

. Such $y(x)$ is then also called '**regular function**'. The kernel $K(x,t)$, a function of two variables is an $\mathfrak{L}_2$ - function if at least one of the following is true:

$$\int_{x=a}^b \int_{t=a}^b |K(x,t)|^2 dx dt < \infty$$

$$\int_{t=a}^b |K(x,t)|^2 dx < \infty$$

$$\int_{x=a}^b |K(x,t)|^2 dt < \infty$$

2.2 Inner Product of two $\mathfrak{L}_2$ functions

The inner product or scalar product (ϕ, ψ) of two complex $\mathfrak{L}_2$ functions ϕ and ψ of a real variable x ; $a \leq x \leq b$ is defined as $(\phi, \psi) = \int_a^b \phi(x)\bar{\psi}(x)dx$
Where $\bar{\psi}(x)$ is the complex conjugate of $\psi(x)$

When $(\phi, \psi) = 0$, or $\int_a^b \phi(x)\bar{\psi}(x)dx = 0$ then ϕ and ψ are called orthogonal to each other.

2.3 Norm of a function

The norm of a complex- function $y(x)$ of a single real variable x is given by $||y(x)|| = \sqrt{\int_a^b y(x)y(\bar{x})dx} = \sqrt{\int_a^b |y(x)|^2 dx}$, where $y(\bar{x})$ represents the complex conjugate of $y(x)$.

The norm of operations between any two functions ϕ and ψ follows Schwarz and Minkowski's triangle inequalities, provided $||\phi \cdot \psi|| \leq ||\phi|| \cdot ||\psi||$ —- Schwarz's Inequality
$||\phi + \psi|| \leq ||\phi|| + ||\psi||$ ——-Triangle Inequality/Minkowski Inequality

2.4 Solution of Integral Equations by Trial Method

A solution of an equation is the value of the unknown function which satisfies the complete equation. The same definition is followed by the solution of an integral equation too. First of all we will handle problems in which a value of the unknown function is given and we are asked to verify whether it's a solution of the integral equation or not. The following example will make it clear:

2.4.1 Example

Show that $y(x) = (1 + x^2)^{-3/2}$ is a solution of

$$y(x) = \frac{1}{1 + x^2} - \int_0^x \frac{t}{1 + x^2} y(t)dt$$

. This is a Volterra's equation of second kind with lower limit $a = 0$ and upper limit being the variable x.

Solution: Given

$$y(x) = \frac{1}{1 + x^2} - \int_0^x \frac{t}{1 + x^2} y(t)dt \ldots (1)$$

where $y(x) = (1 + x^2)^{-3/2} \ldots (2)$
and therefore, $y(t) = (1 + t^2)^{-3/2} \ldots (3)$ (replacing x by t).

The Right Hand Side of (1)

$$= \frac{1}{1 + x^2} - \int_0^x \frac{t}{1 + x^2} y(t)dt$$

$$= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}(1+t^2)^{-3/2} dt$$

[putting the value of $y(t)$ from (3)]

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^x \frac{t}{(1+t^2)^{3/2}} dt$$

since $\frac{1}{1+x^2}$ is independent quantity as the integration is done with respect to t i.e., dt only, therefore $\frac{1}{1+x^2}$ can be excluded outside the integration sign.

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left(\frac{1}{\sqrt{1+x^2}} - 1\right)$$

Since

$$\int_0^x \frac{t}{1+t^{2^{3/2}}} dt$$

$=1-\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-3/2} = y(x)$ =The Left Hand Side of (2) Hence, $y(x) = (1+x^2)^{-3/2}$ is a solution of (1). □ Trial method isn't exactly the way an integral equation can be solved, it is however very important for learning and pedagogy point of views.

Chapter 3

Differential Equations into Integral Equations

A differential equation can be easily converted into an integral equation just by integrating it once or twice or as many times, if needed. Let's start with an example. Let

$$\frac{dy}{dx} + 5y + 1 = 0 \ldots (1)$$

be a simple first order differential equation. We can integrate it one time with respect to x , to obtain

$$\int \frac{dy}{dx} dx + 5 \int y dx + \int 1 \cdot dx = c$$

Or,

$$y + 5 \int y dx + x = c \ldots (2)$$

If we arrange equation (2) in standard integral equation forms, as studied in the first chapter, we get

$$y = (c - x) - 5 \int y dx$$

or,

$$y(x) = (c - x) - 5 \int y(t) dt \ldots (3)$$

We can remove the arbitrary constant c from the above integral equation by applying a boundary condition. For example, if we have

$$y(0) = 1$$

, then it can be easily seen that

$$y(0) = (c - 0) - 5 \int y(0) dt$$

or,

$$c = y(0) + 5 \int y(0) dt$$

$$\Rightarrow c = 1 + 5 \int 1 \cdot dt$$

$$\Rightarrow c = 1 + 5 \int dt \ldots (4)$$

At this instance, we see that if the limits of the integration could have known, the value of c should have been easier to interpret. Still we can convert the given differential equation into integral equation by substituting the value of c in equation (3) above:

$$y(x) = (1 - x + 5 \int dt) - 5 \int y(t)dt$$

$$y(x) = (1 - x) + 5 \int (1 - y(t))dt \ldots (5)$$

Equation(5) is the resulting integral equation converted from equation (1). □

We see that there is only one boundary condition required to obtain the single constant c in First Order differential equation. In the same way, there are two boundary conditions required in a second order differential equation.

Problems in second order differential equation with boundary conditions, are of two types.

Initial Value Problem

For some finite value of variable x, the value of function y and its derivative dy/dx is given in an initial value differential equation problem. For example

$$\frac{d^2y}{dx^2} + ky = tx$$

with

$$y(0) = 2$$

and

$$y'(0) = 5$$

is an initial value problem. Just try to see how, point $x = 0$ is used for both y and y', which is called the initial value of the differential equation. This initial value changes into the lower limit when we try to derive the integral equation. And, also, the integral equation derived from an initial value problem is of Volterra type, i.e., having upper limit as variable x.

Boundary value problem

For different values of variable x, the value of function given in a boundary value condition. For example

$$\frac{d^2y}{dx^2} + ly = mx$$

with

$$y(a) = A$$

and

$$y(b) = B$$

is a boundary value problem. Generally, we chose the lower limit of the integration as zero and integrate the differential equation within limit $(0, x)$. After the boundary values are substituted, we obtain a Fredholm integral equation, i.e., having upper limit as a constant b (say).

All doubts will be cleared by working out the following two examples:

3.1 Converting initial value problem into a Volterra integral equation

Problem 1 : Convert the following differential equation into integral equation:

$$y" + y = 0$$

when

$$y(0) = y'(0) = 0$$

Solution: Given

$$y"(x) + y(x) = 0 \ldots (6)$$

with

$$y(0) = 0 \ldots (7)$$

and

$$y'(0) = 0 \ldots (8)$$

From (1),

$$y"(x) = -y(x) \ldots (9)$$

Integrating (9) with respect to x from 0 to x.

$$\int_0^x y"(x)dx = -\int_0^x y(x)dx$$

$(y'(x))_0^x = -\int 0^x y(x)dx \Rightarrow y'(x) - y'(0) = -\int_0^x y(x)dx$ Since,

$$y'(x) = 0$$

,

$$\Rightarrow y'(x) - 0 = -\int_0^x y(x)dx$$

$$\Rightarrow y'(x) = -\int_0^x y(x)dx \ldots (10)$$

Integrating both sides of (10) with respect to x from 0 to x -

$$\int_0^x y'(x)dx = -\int_0^x \left(\int_0^x y(x)dx\right) dx$$

$$\int_0^x y'(x)dx = -\int_0^x y(x)dx^2$$

$$\Rightarrow y(x)0^x = -\int_0^x y(t)dt^2$$

$$y(x) - y(0) = -\int_0^x (x-t)y(t)dt$$

$$\Rightarrow y(x) - 0 = -\int_0^x (x-t)y(t)dt$$

$$\Rightarrow y(x) = -\int_0^x (x-t)y(t)dt \ldots (11)$$

This equation (11) is the resulting integral equation derived from the given second order differential equation. □

3.2 Converting boundary value problem into a Fredholm integral equation

Example 2:
Reduce the following boundary value problem into an integral equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

with

$$y(0) = 0$$

and

$$y(l) = 0$$

Solution:
Given differential equation is

$$y\text{"}(x) + \lambda y(x) = 0 \ldots (12)$$

with

$$y(0) = 0 \ldots (13)$$

and

$$y(l) = 0 \ldots (14)$$

Since,

$$(12) \Rightarrow y\text{"}(x) = -\lambda y(x) \ldots (15)$$

Integrating both sides of (15) w.r.t. x from 0 to x

$$\int_0^x y\text{"}(x)dx = -\lambda \int_0^x y(x)dx \ldots (16)$$

$$y'(x)0^x = -\lambda \int_0^x y(x)dx$$

$$y'(x) - y'(0) = -\lambda \int_0^x y(x)dx \ldots (17)$$

Let $y'(0) = \textbf{constant} = c$, then

$$y'(x) - c = -\lambda \int_0^x y(x)dx$$

$$y'(x) = c - \lambda \int_0^x y(x)dx \ldots (18)$$

Integrating (18) again with respect to x from 0 to x

$$\int_0^x y'(x)dx = c\int_0^x dx - \lambda \int_0^x \left(\int_0^x y(x)dx \right) dx$$

or,

$$y(x)0^x = cx - \lambda \int_0^x y(x)dx^2$$

$$y(x) - y(0) = cx - \lambda \int_0^x y(t)dt^2$$

Putting $y(0) = 0$

$$y(x) = cx - \lambda \int_0^x (x - t)y(t)dt \ldots (19)$$

Now, putting $x = l$ in (19):

$$y(l) = cl - \lambda \int_0^l (l - t)y(t)dt$$

$$0 = cl - \lambda \int_0^l (l - t)y(t)dt$$

$$c = \frac{\lambda}{l} \int_0^l (l - t)y(t)dt \ldots (20)$$

Putting this value of c in (19), (19) reduces to:

$$y(x) = \frac{\lambda x}{l} \int_0^l (l - t)y(t)dt - \lambda \int_0^x (x - t)y(t)dt \ldots (21)$$

On simplifying (21) we get

$$y(x) = \lambda(\int_0^x \frac{(l - x)t}{l} y(t)dt + \int_x^l \frac{x(l - t)}{l} y(t)dt) \ldots (22)$$

Which is the required integral equation derived from the given differential equation. The solution can also be written as

$$y(x) = \lambda \int_0^l K(x,t)y(t)dt$$

where

$$K(x,t) = \frac{t(l - x)}{l} \qquad \mathbf{0 < t < x}$$

and

$$K(x,t) = \frac{x(l - t)}{l} \qquad \mathbf{x < t < l}$$

□

We can now define a strategy for changing the ordinary differential equations of second order into an integral equation.

Step 1: Write the differential equation and its boundary conditions.
Step 2: Now re-write the differential equation in its normal form, i.e., highest derivatives being on one side and other, all values on the other side. For example, $y" = -\frac{\alpha}{2}xy' + ny$ is the normal form of $2y" + \alpha xy' - 2ny = 0$.
Step 3: Integrate the normal form of the differential equation, from 0 to x. Use applicable rules and formulas to simplify it.
Step 4: If substitutable, substitute the values of the boundary conditions. In boundary value problems, take $y'(0) = c$ a constant.
Step 5: Again integrate, the, so obtained differential-integral equation, within the limits $(0, x)$ with respect to x.
Step 6: Substitute the values of given boundary conditions.
Step 7: Simplify using essential integration rules, change the variable inside the integration sign to t . Use the 'multiple integral' rules to change multiple integral into linear integral, as we discussed in Chapter 1.

Chapter 4

Integral Equations to Differential Equations

The method of converting an integral equation into a differential equation is exactly opposite to what we did in last chapter where we converted boundary value differential equations into respective integral equations. In last workout, initial value problems always ended up as Volterra Integrals and boundary value problems resulted as Fredholm Integrals. In converse process we will get initial value problems from Volterra Integrals and boundary value problems from Fredholm Integral Equations. Also, as in earlier conversion we continuously integrated the differentials within given boundary values, we will continuously differentiate provided integral equations and refine the results by putting all constant integration limits. The above instructions can be practically understood by following two examples. First problem involves the conversion of Volterra Integral Equation into differential equation and the second problem displays the conversion of Fredholm Integral Equation into differential equation.

4.1 Converting Volterra Integral Equation into Ordinary Differential Equation with initial values

Convert

$$y(x) = -\int_0^x (x-t)y(t)dt$$

into initial value problem.

Please note that this was the same integral equation we obtained after converting initial value problem:

$$y" + y = 0$$

when

$$y(0) = y'(0) = 0$$

(See Problem 1 of Chapter 3)

Solution:
We have,

$$y(x) = -\int_0^x (x-t)y(t)dt \ldots (1)$$

Differentiating (1) with respect to x will give

$$y'(x) = -\frac{d}{dx}\int_0^x (x-t)y(t)dt$$

$$\Rightarrow y'(x) = -\int_0^x y(t)dt \ldots (2)$$

Again differentiating (2) w.r.t. x will give

$$y"(x) = -\frac{d}{dx}\int_0^x y(t)dt$$

$$\Rightarrow y"(x) = -y(x) \ldots (3')$$

$$\iff y"(x) + y(x) = 0 \ldots (3)$$

Putting the lower limit $x = 0$ (i.e., the initial value) in equation (1) and (2) will give, respectively the following:

$$y(0) = -\int_0^0 (0-t)y(t)dt$$

$$y(0) = 0 \ldots (4)$$

And,

$$y'(0) = -\int_0^0 y(t)dt$$

$$y'(0) = 0 \ldots (5)$$

These equations (3), (4) and (5) form the ordinary differential form of given integral equation. □

4.2 Converting Fredholm Integral Equation into Ordinary Differential Equation with boundary values

Convert

$$y(x) = \lambda \int_0^l K(x,t)y(t)dt$$

into boundary value problem where

$$K(x,t) = \frac{t(l-x)}{l} \qquad \mathbf{0 < t < x}$$

and

$$K(x,t) = \frac{x(l-t)}{l} \qquad \mathbf{x < t < l}$$

Please see Example 2 of Chapter 3.

Solution:

The given integral equation is

$$y(x) = \lambda \int_0^l K(x,t)y(t)dt \ldots (1)$$

or

$$y(x) = \lambda(\int_0^x \frac{(l-x)t}{l} y(t)dt + \int_x^l \frac{x(l-t)}{l} y(t)dt) \ldots (2)$$

Differentiating (2) with respect to x will give

$$y'(x) = -\frac{\lambda}{l} \int_0^x ty(t)dt + \frac{\lambda}{l} \int_x^l (l-t)y(t)dt \ldots (3)$$

Continued differentiation of (3) will give

$$y"(x) = -\lambda y(x)$$

That's

$$y"(x) + \lambda y(x) = 0 \ldots (4)$$

To get the boundary values, we place x equal to both integration limits in (1) or (2). $x = 0 \Rightarrow$

$$y(0) = 0 \ldots (5)$$

$x = l \Rightarrow$

$$y(l) = 0 \ldots (6)$$

The ODE (4) with boundary values (5) & (6) is the exact conversion of given integral equation. □

For more details see: *gauravtiwari.org*