Everywhere Continuous Non-differentiable Function

Weierstrass had drawn attention to the fact that there exist functions which are continuous for every value of x but do not possess a derivative for any value. We now consider the celebrated function given by Weierstrass to show this fact. It will be shown that if

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) \quad \ldots (1)$$

$$= \cos \pi x + b \cos a\pi x + b^2 \cos a^2 \pi x + \ldots$$

where a is an odd positive integer, $0 < b < 1$ and $ab > 1 + \frac{3}{2}\pi$, then the function f is continuous $\forall x$ but not finitely derivable for any value of x .

G.H. Hardy improved this result to allow $ab \geq 1$.

We have $|b^n \cos(a^n \pi x)| \leq b^n$ and $\sum b^n$ is convergent. Thus, by Wierstrass's M - Test for uniform Convergence the series (1), is uniformly convergent in every interval. Hence f is continuous $\forall x$.

Again, we have

$$\frac{f(x+h) - f(x)}{h}$$

$$= \sum_{n=0}^{\infty} b^n \frac{\cos[a^n \pi(x+h)] - \cos a^n \pi x}{h} \quad \ldots (2)$$

Let, now, m be any positive integer. Also let S_m denote the sum of the m terms and R_m , the remainder after m terms, of the series (2), so that

$$\sum_{n=0}^{\infty} b^n \frac{\cos[a^n\pi(x+h)] - \cos a^n \pi x}{h} = S_m + R_m$$

By Lagrange's mean value theorem, we have

$$\frac{|\cos\left[a^n\pi(x+h)\right] - \cos a^n\pi x|}{|h|}$$

$$= |a^n \pi h \sin a^n \pi(x+\theta h)| \leq a^n \pi |h|$$

$$|S_m| \leq \sum_{n=0}^{m-1} b^n a^n \pi = \pi \frac{a^m b^m - 1}{ab-1} < \pi \frac{a^m b^m}{ab-1} \ .$$

We shall now consider R_m .
So far we have taken h as an arbitrary but we shall now choose it as follows:

We write $a^m x = \alpha_m + \xi_m$, where α_m is the integer nearest to $a^m x$ and $-1/2 \leq \xi_m < 1/2$.
Therefore $a^m(x+h) = \alpha_m + \xi_m + ha^m$. We choose, h , so that $\xi_m + ha^m = 1$
i.e., $h = \dfrac{1-\xi_m}{a^m}$ which $\to 0$ as $m \to \infty$ for $0 < h \leq \dfrac{3}{2a^m}$ $\quad \ldots (3)$

Now,

$$a^n \pi(x+h) = a^{n-m} a^m (x+h)$$

$$= a^{n-m}\pi[(\alpha_m + \xi_m) + (1-\xi_m)] = a^{n-m}\pi(\alpha_m + 1)$$

Thus

$$\cos[a^n\pi(x+h)] = cos[a^{n-m}(\alpha_m - 1)\pi] = (-1)^{\alpha_{m+1}} \ .$$

$\cos(a^n \pi x) = \cos[a^{n-m}(a^m \pi x)]$

$= \cos[a^{n-m}(\alpha_m + \xi_m)\pi]$

$= \cos a^{n-m}\alpha_m\pi \cos a^{n-m}\xi_m\pi - \sin a^{n-m}\alpha_m\pi \sin a^{n-m}\xi_m\pi$

$= (-1)^{\alpha_m} \cos a^{n-m}\xi_m\pi$

for a is an odd integer and α_m is an integer.

Therefore,

$$R_m = \frac{(-1)^{\alpha_m} + 1}{h} \sum_{n=m}^{\infty} b^n [2 + \cos(a^{n-m}\xi_m\pi] \ \ldots (4)$$

Now each term of series in (4) is greater than or equal to 0 and, in particular, the first term is positive, $|R_m| > \frac{b^m}{|h|} > \frac{2a^m b^m}{3} \ \ \ldots (3)$

Thus

$$\left|\frac{f(x+h) - f(x)}{h}\right| = |R_m + S_m| \ \ \geq |R_m| - |S_m| > \left(\frac{2}{3} - \frac{\pi}{ab-1}\right) a^m b^m$$

As $ab > 1 + \frac{3}{2}\pi$

therefore $\left(\frac{3}{2} - \frac{\pi}{ab-1}\right)$ is positive.

Thus we see that when $m \to \infty$ so that $h \to 0$, the expression $\frac{f(x+h) - f(x)}{h}$ takes arbitrary large values. Hence, $f'(x)$ does not exist or is at least not finite.