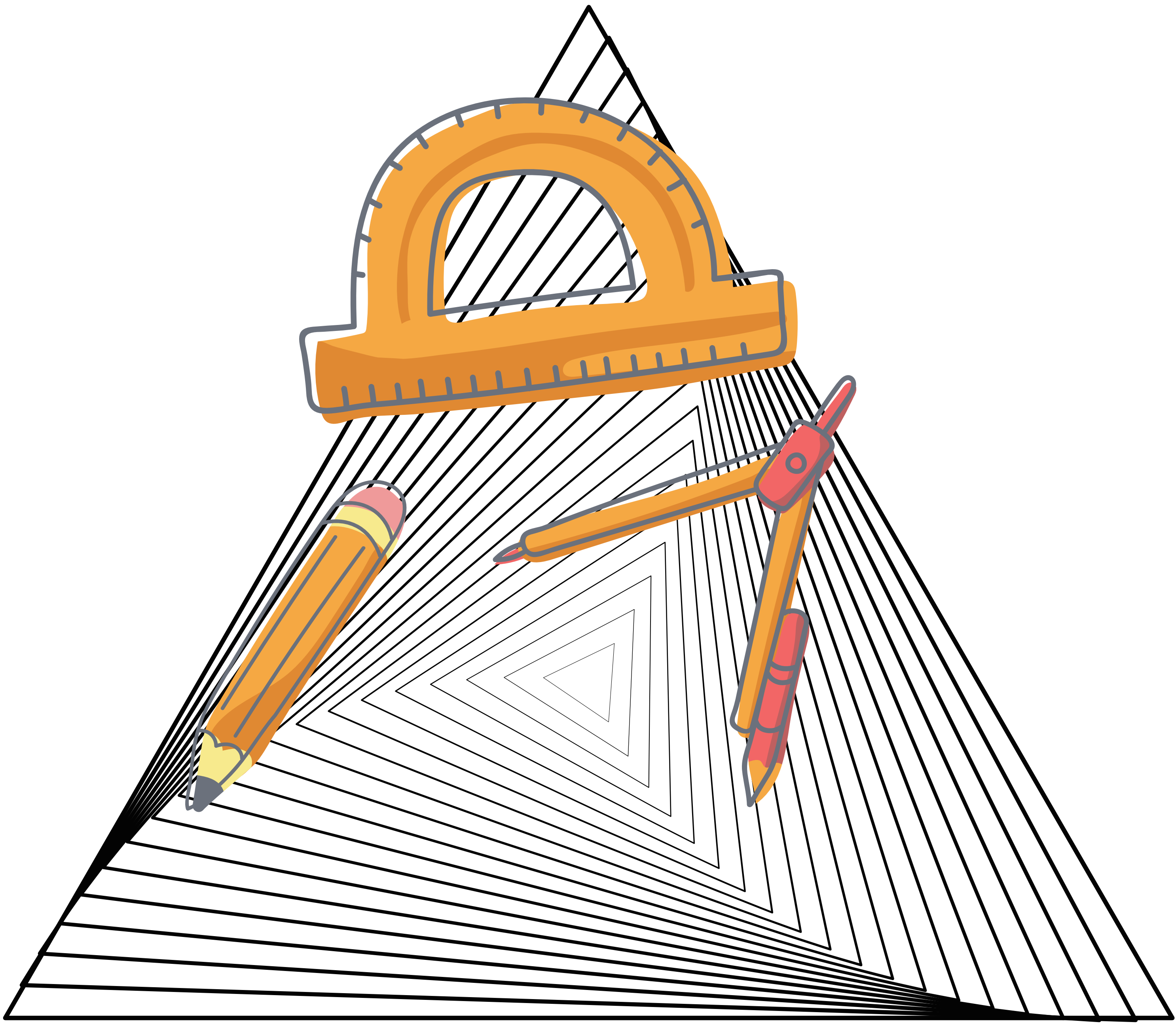


Trigonometric Identities

Trigonometric identities come in handy whenever trigonometric functions are involved in an equation or an expression. These identities are true for every value of variables occurring on both sides of an equation. Geometrically, they involve certain trigonometric functions (such as sine, cosine, and tangent) of one or more angles.

Sine, cosine and tangent are known as the primary trigonometry functions, whereas secant, cosecant, and cotangent are the other three functions. Trigonometric identities are based on all six of these trigonometric functions.

Let us proceed to learn more about trigonometric identities and related formulae.

What are Trigonometric Identities?

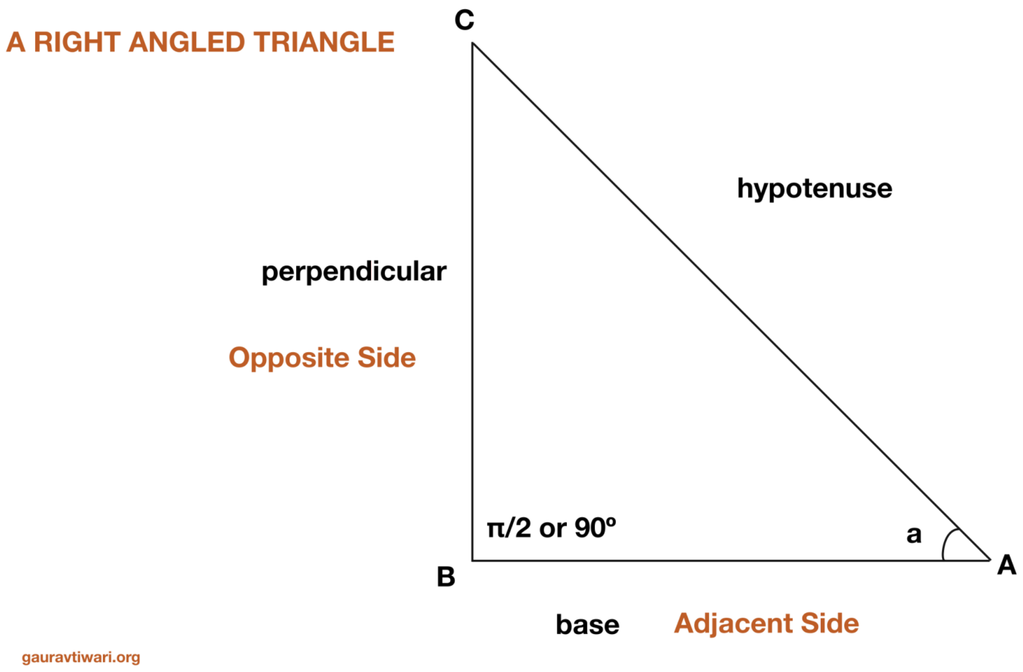
Trigonometric identities are equalities that involve trigonometric functions and hold true for all the given values of variables in the equation. There are various distinct trigonometric identities that involve the side length as well as the angle of a triangle. Keep in mind, though, that these identities hold true only for right-angled triangles.

All trigonometric identities are based on the six trigonometric ratios – sine, cosine, tangent, cosecant, secant, and cotangent. These ratios are all defined using the sides of a right-angled triangle, such as an adjacent side, opposite side, and hypotenuse side.

All the fundamental trigonometric identities are derived from these six trigonometric ratios.

Trigonometric Identities

By using a right-angled triangle as a reference, trigonometric functions and identities are derived as follows:



$$\sin\theta = \frac{\textbf{Opposite side (Perpendicular)}}{\textbf{Hypotenuse}}$$

$$\cos\theta = \frac{\textbf{Adjacent side(Base)}}{\textbf{Hypotenuse}}$$

$$\tan\theta = \frac{\textbf{Opposite side}}{\textbf{Adjacent side}}$$

$$\sec\theta = \frac{\textbf{Hypotenuse}}{\textbf{Adjacent side}}$$

$$\text{cosec}\ \theta = \frac{\textbf{Hypotenuse}}{\textbf{Opposite side}}$$

$$\cot\theta = \frac{\textbf{Adjacent side}}{\textbf{Opposite side}}$$

Ratio trigonometric identities

The trigonometric ratio identities are:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal identities

The reciprocal identities are given as follows:

$$\operatorname{cosec}\,\theta = \frac{1}{\sin\theta}$$

$$\sec\ \theta = \frac{1}{\cos\theta}$$

$$\cot\ \theta = \frac{1}{\tan\theta}$$

$$\sin\ \theta = \frac{1}{\operatorname{cosec}\,\theta}$$

$$\cos\ \theta = \frac{1}{\sec\theta}$$

$$\tan\ \theta = \frac{1}{\cot\theta}$$

All of these identities are taken from a right-angled triangle as well. When the height and base side of the right triangle are known, we can use trigonometric formulae to find out the sine, cosine, tangent, secant, cosecant, and cotangent values. The reciprocal trigonometric identities are derived using the trigonometric functions as well.

Periodicity identities (in radians)

These formulae are also known as co-function identities. We can use them to shift the angles by $\pi/2$, π, 2π etc.

1. $\sin(\pi/2–A) = \cos A$ and $\cos(\pi/2–A) = \sin A$
2. $\sin(\pi/2 + A) = \cos A$ and $\cos(\pi/2 + A) = -\sin A$
3. $\sin(3\pi/2–A) = -\cos A$ and $\cos(3\pi/2–A) = -\sin A$
4. $\sin(3\pi/2 + A) = -\cos A$ and $\cos(3\pi/2 + A) = \sin A$
5. $\sin(\pi–A) = \sin A$ and $\cos(\pi–A) = -\cos A$
6. $\sin(\pi + A) = -\sin A$ and $\cos(\pi + A) = -\cos A$
7. $\sin(2\pi–A) = -\sin A$ and $\cos(2\pi–A) = \cos A$
8. $\sin(2\pi + A) = \sin A$ and $\cos(2\pi + A) = \cos A$

All trigonometric identities are cyclic in nature; in other words, they repeat themselves after this periodicity constant. Keep in mind that the periodicity constant is different for different trigonometric identities.

Cofunction identities (in degrees)

The co-function or periodic identities can be represented in degrees as shown:

$$\sin(90^\circ - x) = \cos x$$

$$\cos(90^\circ - x) = \sin x$$

$$\tan(90^\circ - x) = \cot x$$

$$\cot(90^\circ - x) = \tan x$$

$$\sec(90^\circ - x) = \mathrm{cosec} x$$

$$\mathrm{cosec}(90^\circ - x) = \sec x$$

Sum and difference identities

$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$$

$$\cos(x+y)=\cos(x)\cos(y)–\sin(x)\sin(y)$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\sin(x–y)=\sin(x)\cos(y)–\cos(x)\sin(y)$$

$$\cos(x–y)=\cos(x)\cos(y)+\sin(x)\sin(y)$$

$$\tan(x-y)=\frac{\tan x–\tan y}{1+\tan x\tan y}$$

Double angle identities

$$\sin 2x=2\sin x\cos x=\frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x=\cos^2 x–\sin^2 x=\frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\cos(2x)=2\cos 2(x)-1=1–2\sin 2(x)$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

$$\sec 2x=\frac{\sec^2 x}{2-\sec^2 x}$$

$$\mathrm{cosec}2x=\tfrac{1}{2}\sec x\ \mathrm{cosec}x$$

Triple angle identities

$$\sin 3x = 3\sin x – 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x – \tan^3 x}{1 - 3\tan^2 x}$$

Half angle identities

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x}$$

Product identities

$$\sin x\cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$$

$$\cos x\cos y = \frac{\cos(x+y)+\cos(x-y)}{2}$$

$$\sin x\sin y = \frac{\cos(x-y)+\cos(x+y)}{2}$$

Sum to product identities

$$\sin x + \sin y = 2\sin\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$$

$\sin x - \sin y = 2\cos\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$

$\cos x + \cos y = 2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$

$\cos x + \cos y = 2\sin\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$

Pythagorean trigonometric identities

There are three Pythagorean trigonometric identities based on the right-triangle theorem or Pythagoras theorem:

$\sin^2 a + \cos^2 a = 1$

$1 + \tan^2 a = \sec^2 a$

$\mathrm{cosec}^2 a = 1 + \cot^2 a$

Trigonometric identities of opposite angles

Below is the list of opposite-angle trigonometric identities:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$

$\sec(-\theta) = \sec\theta$

$\mathrm{cosec}\ (-\theta) = -\mathrm{cosec}\ \theta$

Trigonometric identities of complementary angles

In geometry, two angles are said to be complementary if their sum is equal to 90 degrees. In this section, we will learn about the trigonometric identities for complementary angles.

$$\sin(90^\circ – \theta) = \cos\theta$$

$$\cos(90^\circ – \theta) = \sin\theta$$

$$\tan(90^\circ – \theta) = \cot\theta$$

$$\cot(90^\circ – \theta) = \tan\theta$$

$$\sec(90^\circ – \theta) = \text{cosec }\theta$$

$$\text{cosec }(90^\circ – \theta) = \sec\theta$$

Trigonometric identities of supplementary angles

Two angles are said to be supplementary if their sum is equal to 90 degrees. In this section, we will learn about the trigonometric identities for supplementary angles.

$$\sin(180^\circ - \theta) = \sin\theta$$

$$\cos(180^\circ - \theta) = -\cos\theta$$

$$\text{cosec }(180^\circ - \theta) = \text{cosec }\theta$$

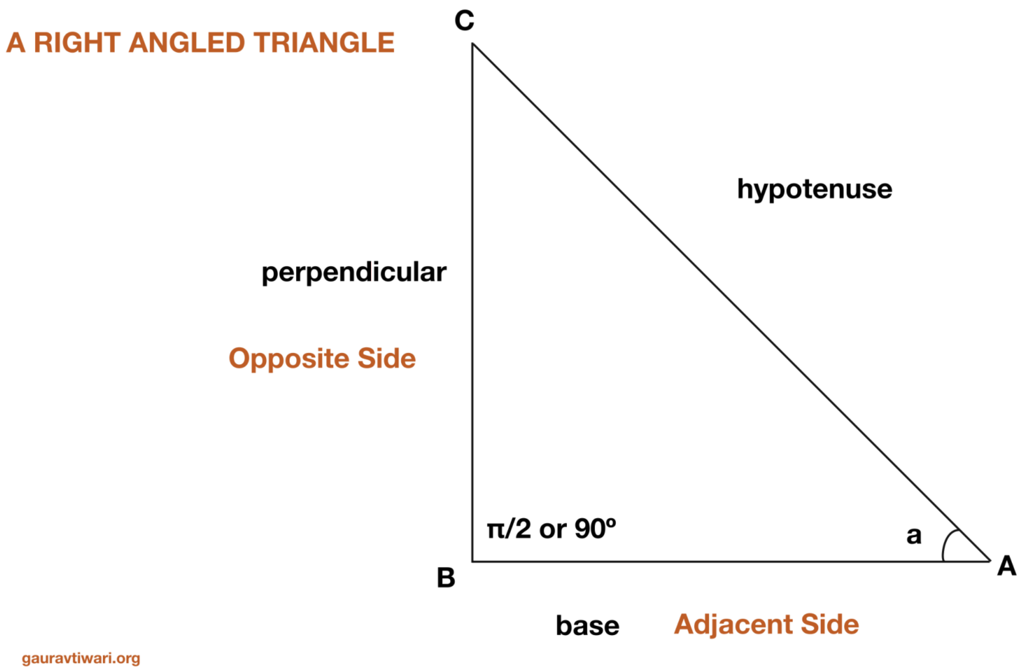
$$\sec(180^\circ - \theta) = -\sec\theta$$

$$\tan(180^\circ - \theta) = -\tan\theta$$

$$\cot(180^\circ - \theta) = -\cot\theta$$

Proofs of trigonometric identities

An equation that involves trigonometric ratios of an angle represents a trigonometric identity. In this section, we will look at the fundamental trigonometric identities and their proofs. Consider a right-angled $\triangle ABC$, which is right-angled at B, as shown in the figure below.



Applying the Pythagoras theorem for the given triangle, we get:

(hypotenuse)^2 = (base)^2 + (perpendicular)^2

$$AC^2 = AB^2 + BC^2 \ldots (1)$$

Let us proceed to prove the three commonly used Pythagoras trigonometric identities.

Trigonometric identity 1

Now, on dividing each term of equation (1) by AC^2, we get:

$$\frac{AC^2}{AC^2} = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$$

$$\implies \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

$$\implies (\frac{AB}{AC})^2 + (\frac{BC}{AC})^2 = 1 \ldots (2)$$

We know that $\frac{AB}{AC}$ is $\cos a$ and $\frac{BC}{AC}$ is $\sin a$.

Therefore, we can write equation (2) as:

$$\sin^2 a + \cos^2 a = 1$$

This identity is valid for angles $0 \leq a \leq 90^\circ$.

Trigonometric identity 2

Now, on dividing the equation (1) by AB^2, we get:

$$\frac{AC^2}{AB^2} = \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2}$$

$$\implies \frac{AC^2}{AB^2} = 1 + \frac{BC^2}{AB^2}$$

$$\implies (\frac{AC}{AB})^2 = 1 + (\frac{BC}{AB})^2 \ldots (3)$$

We know that $\frac{AC}{AB}$ is inverse of $\cos a$ or $\sec a$ and $\frac{BC}{AB}$ is $\tan a$.

On replacing the values of AC/AB and BC/AB in equation (3), we get:

$$1 + \tan^2 a = \sec^2 a \ldots (4)$$

We know that tan a is not defined for $a = 90^\circ$, therefore, the trigonometric identity 2 we obtained above holds true for $0 \leq a < 90^\circ$.

Trigonometric identity 3

Similarly by dividing the equation (1) by BC^2, we can get:

$$\operatorname{cosec}^2 a = 1 + \cot^2 a \ldots (5)$$

Since $\operatorname{cosec} a$ and $\cot a$ are not defined for $a = 0^\circ$, the trigonometric identity (5) we obtained above is true for all the values of 'a' except at a = 0°. Therefore, it is true for all such that: $0^\circ < a \leq 90^\circ$.

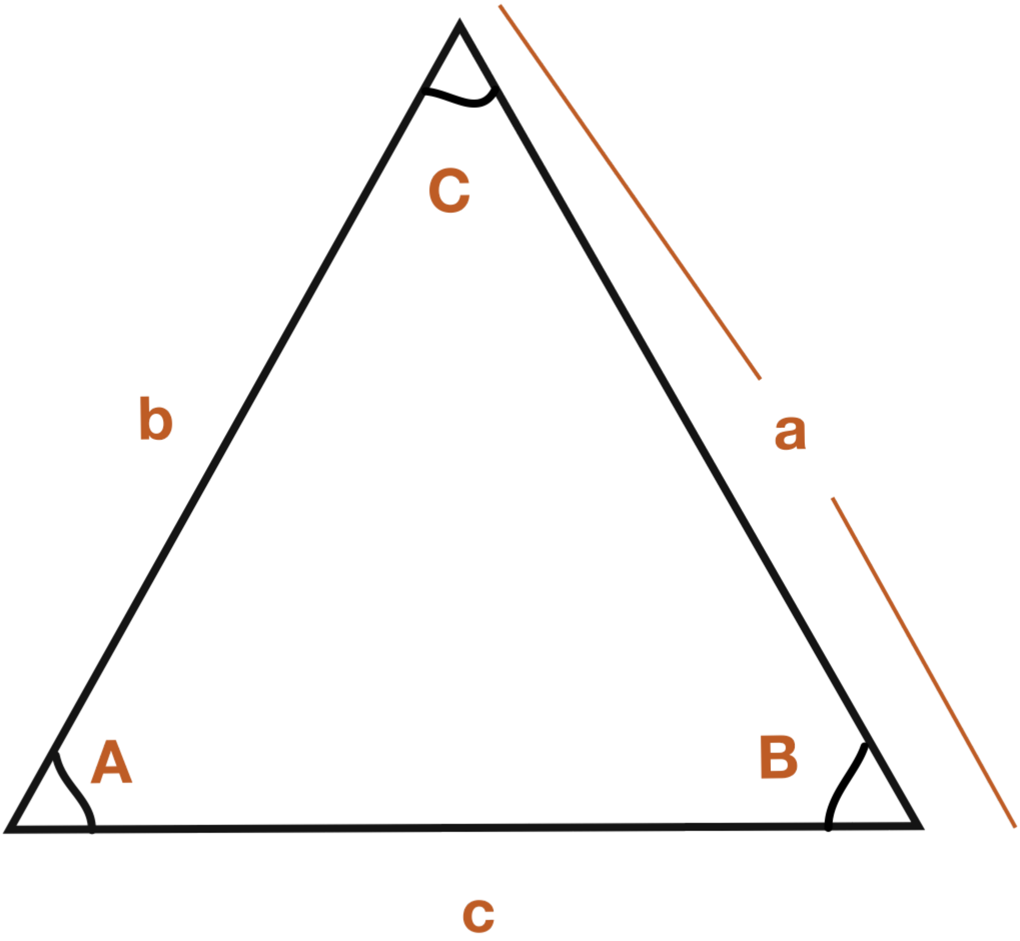
Triangle identities (the Sine, Cosine, and Tangent rules)

Identities or equations that are applicable for all triangles and not just for right-angled triangles, are known as triangle identities. These identities include the:

- Sine law
- Cosine law
- Tangent law

Also see: Triangle Inequality

Sine Law



If A, B and C are the vertices of a triangle and a, b and c are the respective sides, then according to the sine law or sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Law

According to the cosine law,

$$c^2 = a^2 + b^2 – 2ab \cos C$$

or, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Tangent Law

According to the tangent law,

$$\frac{a - b}{a + b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$